

Neighbourhoods of Isolated Horizons and their stationarity

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A distinguished (invariant) Bondi-like coordinate system is defined in the spacetime neighbourhood of a non-expanding horizon of arbitrary dimension via geometry invariants of the horizon. With its use, the radial expansion of a spacetime metric about the horizon is provided and the free data needed to specify it up to given order are determined in spacetime dimension 4. For the case of an electro-vacuum horizon in 4-dimensional spacetime the necessary and sufficient conditions for the existence of a Killing field at its neighbourhood are identified as differential conditions on the horizon data and data on null surface transversal to the horizon.

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I. INTRODUCTION

Systematic studies of black holes in various approaches to quantum gravity as well as accurate description of the dynamical evolution of these exotic objects require a quasi-local description formalism – where a black hole can be treated as “object in the lab” and the global spacetime structure of the universe far away from it can be neglected. Among several approaches to construct such formalism [1, 2] one of considerable success is the theory of Isolated Horizons [3–5]. This approach was originally inspired by ideas of Pejterski and Newman [1], next shaped into a solid formalism by Ashtekar [6] and subsequently developed by many researchers. Its main feature is the representation of a black hole in equilibrium through its surface – the non-expanding (or isolated) horizon – a null cylinder of codimension 1 and of compact spatial slices embedded in a Lorentzian spacetime. The black hole is characterized by the geometry (and possibly matter fields) data on this surface only. Both geometry aspects [6, 7] and the mechanics [8, 9] have been systematically studied in spacetime dimension 4 and then extended to general dimension [10–12], the latter including in particular asymptotically Anti-deSitter spacetimes [13]. Also various matter content at the horizon has been considered [14] as well as the properties of symmetric horizons [15] and their relation with standard black hole solutions [16, 17]. The formalism has been further extended to the non-equilibrium situations through the *Dynamical Horizons* [18] (see also [4]). It is vastly applied in numerical relativity (see for example [19]) as well as in black hole description in loop quantum gravity [20] – especially as the basis for entropy calculations (see for example [21]). The extension to this formalism has found applications also in supergravity [22] and string theory inspired gravity [23].

The quasi-locality of the theory is a great advantage, as only the geometry objects at the horizon are relevant in the description, however for this very reason one misses the information about the black hole neighbourhood. However, the success of the formalism of *near horizon geometries* [24] shows clearly, that there is a strong demand for any black hole description method to be able to also “handle” its neighbourhood, a feature particularly relevant for the studies of black hole spacetimes in context of AdS/CFT correspondence [25] and in numerical probing of the late stages of black hole mergers.

This article is dedicated to supplying such extension within the isolated horizon formalism. Its main ideas have been originally published in [26]. The principal part of presented work is providing the convenient way of describing the spacetime geometry in the neighbourhood of the horizon through the *Bondi-like* coordinate system originally introduced in [27]. Here, this construction is extended to arbitrary spacetime dimension and horizon spatial slice topology and its properties are studied. It is shown to be relatively convenient in use, providing for example a well defined invariant radial spacetime metric expansion about the horizon. It provides a solid frame for addressing the questions like the conditions for the existence of a Killing vector fields in the horizon neighbourhood, which problem is studied here in detail in context of the electro-vacuum black hole in 4-dimensional spacetime.

The paper is organized as follows: We start in section II with short introduction of the geometric structure of general non-expanding horizons in arbitrary dimension and discuss the definition and those properties of the horizon symmetries, which will be needed in the further studies of Killing horizons. Next, in section III, still keeping the

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general dimension, we construct the Bondi-like coordinate system which later will provide the basis to describe the structures at the neighbourhood. In the same section we study the properties and form of Killing fields possibly present there. The material of these two sections is then used in section IV, where we restrict the studies to the horizons in 4-dimensional electrovac spacetime, introducing in particular convenient Newman-Penrose null frame (consistent with Bondi-like coordinate system) for any matter type and studying the initial value problem for the “radial” metric expansion. The interest is then restricted just to horizons embeddable in the electrovac spacetime. There the Maxwell evolution equations change the mentioned initial value problem, making it stronger (less initial data needed). This structure is then applied in section V, where we formulate the conditions necessary and sufficient for the non-expanding horizon to be Killing horizon. Analogous conditions for several classes of Killing fields are derived in section VI. We summarize the results in the concluding section VII. In order to not disrupt the reasoning flow some more complicated proofs as well as the definition of Newman-Penrose frame have been moved to appendices A through D.

II. GEOMETRY OF A NON-EXPANDING HORIZON

A. Non-expanding horizons

In this section we introduce the notation, recall the definition and outline the properties of non-expanding horizons in n dimensions [3, 7, 11, 15]. We often use the abstract index notation and the following convention: the n -spacetime indexes are denoted by α, \dots, ν , every $n - 1$ dimensional vector spaces indexes are denoted by a, \dots, d and each $n - 2$ vector space indexes are denoted by A, \dots, D .

1. Definition, the induced degenerate metric, 2-volume and Hodge *

We start our consideration with an $(n - 1)$ -dimensional null surface Δ embedded in an n -dimensional, time oriented spacetime \mathcal{M} . In Section IV and further $n = 4$. The spacetime metric tensor $g_{\mu\nu}$ of the signature $(-, +, \dots, +)$ is assumed to satisfy the Einstein field equations (possibly with matter and cosmological constant). Throughout the paper, we assume that all the matter fields possibly present at the surface Δ satisfy the following:

Condition II.1. (*Stronger Energy Condition*) *At every point of Δ , for every future oriented null vector ℓ tangent to Δ , the vector $-T^\mu{}_\nu \ell^\nu$ is causal and future oriented, where $T_{\mu\nu}$ is the energy-momentum tensor.*

We denote the degenerate metric tensor induced at Δ by q_{ab} . The sub-bundle of the tangent bundle $T(\Delta)$ defined by the null vectors is denoted by L and referred to as the null direction bundle. Given a vector bundle P , the set of sections will be denoted by $\Gamma(P)$.

To recall the definition of a non-expanding horizon consider the metric tensor q_{AB} induced by the tensor q_{ab} in the fibers of the quotient bundle $T(\Delta)/L$. Denote the inverse metric tensor defined in the fibers of the dual bundle $(T(\Delta)/L)^*$ by $q^{AB}(x)$ (that is $q_{AB}q^{BC} = \delta_A^C$).

We will be assuming that Δ is a non-expanding horizon, in the following sense:

Definition II.2. *A null submanifold Δ of codimension 1 embedded in spacetime satisfying the Einstein field equations is called a non-expanding horizon (NEH) if:*

1. *for every point $x \in \Delta$ for every null vector ℓ^a tangent to Δ at x ,*

$$q^{AB} \mathcal{L}_\ell q_{AB} = 0, \quad (2.1)$$

and

2. *Δ has the product structure $\hat{\Delta} \times \mathcal{I}$, that is there is an embedding*

$$\hat{\Delta} \times \mathcal{I} \rightarrow \mathcal{M} \quad (2.2)$$

such that:

- (i) *Δ is the image,*
- (ii) *$\hat{\Delta}$ is an $n - 2$ dimensional compact and connected¹ manifold (referred to as the horizon base space),*

¹ In the case $\hat{\Delta}$ is not connected all the otherwise global constants (like surface gravity) remain constant only at maximal connected components of the horizon.

(iii) \mathcal{I} is an open interval,

(iv) for every maximal null curve in Δ there is $\hat{x} \in \hat{\Delta}$ such that the curve is the image of $\{\hat{x}\} \times \mathcal{I}$.

Each (non-vanishing in a generic set) null vector field ℓ defined in Δ (a section of L) determines a function $\kappa^{(\ell)}$ referred to as surface gravity² of ℓ , such that

$$\ell^\mu \nabla_\mu \ell^\nu = \kappa^{(\ell)} \ell^\nu . \quad (2.3)$$

In particular, given a NEH there always exists a nowhere vanishing null vector field ℓ_o^a of the identically vanishing surface gravity, $\kappa^{(\ell_o)} = 0$. One can also choose a null vector field ℓ^a of $\kappa^{(\ell)}$ being an arbitrary constant,³

$$\kappa^{(\ell)} = \text{const} . \quad (2.4)$$

That vector field ℓ^a can vanish in a harmless (for our purposes) way on an $(n-2)$ -dimensional section of Δ only.

From Stronger Energy Condition II.1 it follows in particular that $T_{\ell\ell} \geq 0$. This in turn implies via the generalized Raychaudhuri equation that the flow $[\ell]$ preserves the degenerate metric q

$$\mathcal{L}_\ell q_{ab} = 0 , \quad (2.5)$$

and using again the Stronger Energy Condition it can be shown that at every point p of Δ and for every $X \in T_p \Delta$

$$T_{a\beta} X^a \ell^\beta = {}^{(n)}\mathcal{R}_{a\beta} X^a \ell^\beta = 0 . \quad (2.6)$$

The property (2.5) above combined with $\ell^a q_{ab} = 0$ means that q_{ab} is the pullback of a certain metric tensor field \hat{q}_{AB} defined on $\hat{\Delta}$. The horizon base space $\hat{\Delta}$ can be identified with the space of null curves tangent to Δ . Its manifold structure is unique. The pullback map is defined by the natural (and also unique) projection,

$$\pi : \Delta \rightarrow \hat{\Delta} , \quad q_{ab} = (\pi^* \hat{q})_{ab} . \quad (2.7)$$

The pullback of the base space $\hat{\Delta}$ 2-volume form $\hat{\eta}_{AB}$,

$$\pi^* \hat{\eta}_{ab} =: \eta_{ab} \quad (2.8)$$

defines the canonical area 2-form on Δ and its restriction η_{AB} to a two form in $T(\Delta)/L$. η_{AB} is used to define the horizon Hodge dualization $\star_\Delta : (T(\Delta)/L)^* \rightarrow (T(\Delta)/L)^*$,

$$\star_\Delta v_A := \eta_{AB} q^{BC} v_C . \quad (2.9)$$

2. The induced covariant derivative and the rotation 1-form

It can be shown by using (2.5), that the space time covariant derivative ∇_α determined by the metric tensor $g_{\alpha\beta}$ preserves the tangent bundle $T(\Delta)$. Indeed, for every pair of vector fields $X, Y \in \Gamma(T(\Delta))$,

$$\nabla_X Y \in \Gamma(T(\Delta)) . \quad (2.10)$$

Therefore, there exists in $T(\Delta)$ an induced covariant derivative D_a , such that for every pair of vector fields $X, Y \in \Gamma(T(\Delta))$ the following holds

$$D_X Y^a := \nabla_X Y^a . \quad (2.11)$$

Its action on covectors, sections of the dual bundle $T^* \Delta$ is determined by the Leibniz rule. Together with the induced metric the covariant derivative constitutes the *geometry of a NEH* (q_{ab}, D_a) .

The connection D_a preserves in particular the null direction bundle L , thus for every $\ell \in \Gamma(L)$ the derivative $D_a \ell^b$ is proportional to ℓ^b itself,

$$D_a \ell^b = \omega^{(\ell)}{}_a \ell^b , \quad (2.12)$$

² We use dimensionless coordinates in spacetime, therefore our surface gravity is also dimensionless.

³ The first one, ℓ_o can be defined by fixing appropriately affine parameter v at each null curve in Δ . Then, the second vector field is just $\ell = \kappa^{(\ell)} v \ell_o$.

where $\omega^{(\ell)}_a$ is a 1-form defined uniquely on this subset of Δ on which $\ell \neq 0$ is defined. We call $\omega^{(\ell)}_a$ the *rotation 1-form potential* (see [7, 11]).

The evolution of $\omega^{(\ell)}_a$ along the null flow on Δ is responsible for the 0th Law of the non-expanding horizon thermodynamics: the rotation 1-form potential $\omega^{(\ell)}$ and surface gravity of ℓ , related to $\omega^{(\ell)}_a$ via

$$\kappa^{(\ell)} = \ell^a \omega^{(\ell)}_a \quad (2.13)$$

satisfy the following constraint:

$$\mathcal{L}_\ell \omega^{(\ell)}_a = D_a \kappa^{(\ell)} \quad (2.14)$$

implied by (2.6). This tells us in particular, that there is always a choice of the section ℓ of the null direction bundle L such that $\omega^{(\ell)}$ is Lie dragged by ℓ . Indeed, we can always find a non-trivial section ℓ of L such that $\kappa^{(\ell)}$ is constant (0 for ℓ defined by an affine parametrization of the null geodesics tangent to Δ). Throughout the remaining part of the article we will restrict our consideration to fields $\ell \in \Gamma(L)$ of such class, or equivalently satisfying

$$\mathcal{L}_\ell \omega^{(\ell)} = 0. \quad (2.15)$$

Upon rescalings $\ell \mapsto \ell' = f\ell$ (where f is a real function defined at Δ) of a section ℓ^a of L the rotation 1-form changes as follows

$$\omega^{(\ell')}_a = \omega^{(\ell)}_a + D_a \ln f. \quad (2.16)$$

The requirement that both $\kappa^{(\ell)}$ and $\kappa^{(\ell')}$ are constants restricts the form of f to the following one

$$f = \begin{cases} B e^{-\kappa^{(\ell)} u + \frac{\kappa^{(\ell')}}{\kappa^{(\ell)}}} & \kappa^{(\ell)} \neq 0 \\ \kappa^{(\ell')} u - B & \kappa^{(\ell)} = 0 \end{cases}, \quad (2.17)$$

where u is any function defined on Δ such that

$$\ell^a D_a u = 1, \quad (2.18)$$

and B is an arbitrary function constant along null geodesics of Δ .

3. The constraints

The non-expanding horizon geometry (q_{ab}, D_a) is constrained by the Einstein equations. We have already used some of them above. A *complete* set of the constraints on horizon geometry (q_{ab}, D_a) is encoded in the following identity

$$[\mathcal{L}_\ell, D_a] X^b = \ell^b X^c \left[D_{(a} \omega^{(\ell)}_{c)} + \omega^{(\ell)}_a \omega^{(\ell)}_c + \frac{1}{2} \left({}^{(n)}\mathcal{R}_{ac} - \pi^* {}^{(n-2)}\mathcal{R}_{ac} \right) \right] =: \ell^b N_{ac}^{(\ell)} X^c, \quad (2.19)$$

which holds for every $\ell \in \Gamma(L)$ and $X \in \Gamma(T(\Delta))$, where ${}^{(n-2)}\mathcal{R}_{AB}$ is the Ricci tensor of the metric tensor \hat{q}_{AB} induced in base space $\hat{\Delta}$ whereas ${}^{(n)}\mathcal{R}_{ac}$ is the pullback to Δ (by the embedding map $\Delta \rightarrow \mathcal{M}$) of the spacetime Ricci tensor. The constraints are given by replacing ${}^{(n)}\mathcal{R}_{ac}$ with $8\pi G(T_{ac} - \frac{1}{2}Tq_{ac})$, where the pullback onto Δ of the stress energy tensor T_{ac} satisfies on Δ the condition

$$\ell^a T_{ab} = 0 \quad (2.20)$$

(see 2.6). In particular, the 0th law (2.14) is given by contracting (2.19) with a null vector ℓ . The remaining constraints determine the evolution of some other components of D_a along the null geodesics tangent to ℓ provided the energy momentum tensor is given. In the Einstein vacuum or Einstein Maxwell vacuum cases the constraints are solved explicitly [11].

Throughout this paper we are making a stronger assumption, namely that on Δ in addition to (2.20) the pull-back onto Δ (by the embedding map $\Delta \rightarrow \mathcal{M}$) of the energy-momentum tensor is Lie dragged by (any)⁴ null vector field ℓ tangent to Δ

$$\mathcal{L}_\ell T_{ab} = 0. \quad (2.21)$$

For the electromagnetic field considered later in this paper this condition is a consequence of the Einstein-Maxwell equations, therefore it is satisfied automatically. For the time being, however, we just assume it is true.

⁴ In fact, if this condition is satisfied by any given non-vanishing ℓ^a then it is satisfied by every ℓ^a null and tangent to Δ .

4. The invariants

For every non-expanding horizon Δ which satisfies the assumptions made in the previous subsection, the geometry (q, D) is analytic along the null geodesics in any affine coordinate. A given non-expanding horizon Δ can be incomplete in that coordinate, however one can consider its (non-embedded) maximal analytic extension $\bar{\Delta}$ endowed with the analytic extension (\bar{q}, \bar{D}) of the geometry. This is what we do in this section. We will be using the same notation as above however we will mark all the symbols referring to $\bar{\Delta}$ by the extra $\bar{\cdot}$. Finally, all the invariant structures introduced on $\bar{\Delta}$ determine unique restriction to an original, given unextended Δ .

Thus far we reduced the freedom in choice of a null vector field $\bar{\ell}$ tangent to $\bar{\Delta}$ to vector fields which satisfy (2.4) defined up to the transformations given by (2.17). The freedom can be further reduced by imposing some condition on the tensor $N_{ab}^{(\bar{\ell})}$ (2.19). We also notice, that from (2.14, 2.15) it follows that

$$\bar{\ell}^a N_{ab}^{(\bar{\ell})} = 0, \quad (2.22)$$

hence this tensor defines a unique tensor $N_{AB}^{(\bar{\ell})}$ in the fibers of $T(\bar{\Delta}) / \bar{L}$.

Let us now introduce a specific class of $\bar{\ell}$, namely:

Definition II.3. *A natural vector field $\bar{\ell}$ in $\bar{\Delta}$, is a tangent null vector field (that is a section of \bar{L}) non-vanishing on a generic (open and dense) subset of $\bar{\Delta}$ which satisfies the following conditions*

$$\kappa^{(\bar{\ell})} = 1, \quad \bar{q}^{AB} N_{AB}^{(\bar{\ell})} = 0. \quad (2.23)$$

The generic existence and uniqueness of the natural vector field in a given $\bar{\Delta}$, was shown in [11] (see Eq. (6.22) therein and the following paragraph). We review the argumentation proving these properties further below. In the case when the natural vector field is unique we will call it *the invariant vector field of the $\bar{\Delta}$ geometry*.

Given a non-zero null vector field $\bar{\ell}$ in $\bar{\Delta}$ such that $\kappa^{(\bar{\ell})} = \text{const} \neq 0$, a unique foliation by spacelike, $n - 2$ dimensional sections of $\bar{\Delta}$ can be fixed, by using the rotation 1-form potential $\omega^{(\bar{\ell})}$. In particular, one can choose a section $\bar{\sigma} : \hat{\Delta} \rightarrow \bar{\Delta}$ of $\bar{\pi} : \bar{\Delta} \rightarrow \hat{\Delta}$ such that the pullback $\bar{\sigma}^* \omega^{(\bar{\ell})}$ is divergence free,

$$d \star \bar{\sigma}^* \omega^{(\bar{\ell})} = \bar{q}^{AB} \bar{D}_A \bar{\sigma}^* \omega^{(\bar{\ell})}{}_B = 0. \quad (2.24)$$

Those sections of $\bar{\Delta}$ are called *good cuts* [7] and are defined uniquely modulo the action of the flow of the vector field $\bar{\ell}$. They set a foliation of $\bar{\Delta}$. If $\bar{\ell}$ is the invariant vector field of the $\bar{\Delta}$ geometry, then the corresponding good cut foliation will be called the *invariant foliation of the $\bar{\Delta}$ geometry*. One has to be aware though, that whereas the slices of the invariant foliation of $\bar{\Delta}$ are diffeomorphic to the base space $\hat{\Delta}$, the restriction of a slice of $\bar{\Delta}$ to Δ may be a proper subset of that slice. In other words, the slices of Δ may be non-global sections of $\pi : \Delta \rightarrow \hat{\Delta}$. Given a null vector field $\bar{\ell}$ and a foliation there is a unique differential 1-form $\bar{n}_a \in \Gamma(T^*(\bar{\Delta}))$ orthogonal to the leaves of the foliation and normalized by

$$\bar{n}_a \bar{\ell}^a = -1. \quad (2.25)$$

We will call \bar{n}_a the *invariant co-vector of the $\bar{\Delta}$ geometry* if $\bar{\ell}$ and the foliation are, respectively, the invariant vector field of the $\bar{\Delta}$ geometry and the invariant foliation of $\bar{\Delta}$. Finally, the invariant vector field, foliation and covector field of $(\bar{\Delta}, \bar{q}_{ab}, \bar{D}_a)$ are restricted to a given NEH (Δ, q_{ab}, D_a) the geometry of which was the starting point of an extension $(\bar{\Delta}, \bar{q}_{ab}, \bar{D}_a)$. The uniqueness of this extension guaranties the uniqueness of the resulting invariant structures defined on (Δ, q_{ab}, D_a) .

The natural vector field defined above exists and is unique for a certain class of NEH geometries, denoted as *generic* and defined as follows: Let (Δ, q_{ab}, D_a) be a NEH. Choose any null vector field ℓ' such that $\kappa^{(\ell')} = \text{const}$ and any global section $\bar{\sigma} : \hat{\Delta} \rightarrow \bar{\Delta}$. The existence of a natural vector field depends on the invertability of a certain operator introduced in [11]. It involves the following ingredients defined on $\hat{\Delta}$: the induced metric tensor \hat{q}_{AB} (2.7), the corresponding covariant derivative \hat{D}_A and the Ricci scalar ${}^{(n-2)}\hat{\mathcal{R}}$, the pullback $\hat{\omega}_A$ of the rotation 1-form potential $\omega^{(\ell')}_a$ by the section $\bar{\sigma}$, the trace \hat{T} of the pullback \hat{T}_{AB} of the energy-momentum tensor $T_{\alpha\beta}$, and $T := T^\alpha{}_\alpha$. The operator is

$$\hat{M} = \hat{D}^A \hat{D}_A + 2\hat{\omega}^A \hat{D}_A + \hat{\omega}_A \hat{\omega}^A + \hat{D}_A \hat{\omega}^A + 4\pi G \left(\hat{T} - T - {}^{(n-2)}\hat{\mathcal{R}} \right). \quad (2.26)$$

If the kernel of this operator is trivial, then there exists exactly one natural vector field ℓ . Suppose, for given data the operator has a nontrivial kernel. Then, the operator corresponding to a new, gently perturbed data, say

$T'_{\alpha\beta} = T_{\alpha\beta} + \delta\Lambda g_{\alpha\beta}$, has trivial kernel for a non-zero perturbation $|\delta\Lambda| \in (0, \epsilon)$. That shows the genericity of the existence and uniqueness of the natural vector. It is important to point out, that the operator \hat{M} defined above was constructed with use of more data, than (q_{ab}, D_a) , since the objects depending on the choice of a null vector field ℓ' and section of Δ are present on the right hand side of (2.26). However, the dimension of the kernel of this operator is independent of those choices.

If the kernel is nontrivial, on the other hand, then Δ either admits no natural vector field or admits more than one. The latter happens for example if (Δ, q_{ab}, D_a) has 2-dimensional group of null symmetries [15].

In conclusion:

Definition II.4. A NEH (Δ, q_{ab}, D_a) is invariant-generic if it defines the invariant vector field.

In an invariant-generic case, we have defined the following unique structures of (Δ, q_{ab}, D_a) :

- an invariant tangent null vector field ℓ ,
- an invariant foliation by good cuts –sections of $\pi : \Delta \rightarrow \hat{\Delta}$ – preserved (locally) by the flow of ℓ ,
- a function $v : \Delta \rightarrow \mathbb{R}$, constant on the leaves of the foliation and such that

$$\ell^a D_a v = 1, \quad (2.27)$$

invariant up to $v \mapsto v + v_o$, $v_o \in \mathbb{R}$,

- an invariant covector field

$$n = -dv \quad (2.28)$$

orthogonal to the leaves of the invariant foliation.

The reason for the name ‘invariant’ is that given two NEHs (Δ, q_{ab}, D_a) and (Δ', q'_{ab}, D'_a) related by an isomorphism $\phi : \Delta \rightarrow \Delta'$, the corresponding invariants are mapped to each other by ϕ_* , ϕ , ϕ^* and ϕ^* respectively. One has to remember though, that the slices of Δ may be non-global sections of $\pi : \Delta \rightarrow \hat{\Delta}$.

B. Symmetric NEH

1. Definitions, known results

Given a non-expanding horizon Δ , an *infinitesimal symmetry* of it is a non-trivial vector field, $X \in \Gamma(T(\Delta))$ such that

$$\mathcal{L}_X q_{ab} = 0, \quad \text{and} \quad [\mathcal{L}_X, D_a] = 0. \quad (2.29)$$

Each Killing field defined in a spacetime neighbourhood of a NEH Δ and tangent to Δ induces an infinitesimal symmetry of Δ . Therefore, recalling the properties of symmetric NEHs (studied in [15]) is a natural starting point for the current paper. Below we will briefly list those of their properties which are relevant for our studies.

Every infinitesimal symmetry X preserves the null direction, that is for every null vector field $\ell \in \Gamma(T(\Delta))$,

$$[X, \ell] = f\ell, \quad f : \Delta \rightarrow \mathbb{R}. \quad (2.30)$$

Due to this property the projection $\pi : \Delta \rightarrow \hat{\Delta}$ pushes X forward to a uniquely defined vector field on $\hat{\Delta}$, that is there is a vector field $\hat{X} \in \Gamma(T(\hat{\Delta}))$ such that

$$\pi_* X = \hat{X}. \quad (2.31)$$

The vector field \hat{X} is a Killing vector of the geometry $(\hat{\Delta}, \hat{q}_{ab})$.

Every infinitesimal symmetry X defines a unique analytic extension \bar{X} to the maximal analytic extension $\bar{\Delta}$ of Δ . The vector field \bar{X} defines a global flow on $\bar{\Delta}$, and the flow preserves the geometry (\bar{q}, \bar{D}) [15]. Using this property, in this subsection we will consider symmetric maximal analytic extensions of the NEHs.

We distinguish several classes of infinitesimal symmetries. One of them is *null infinitesimal symmetry*, corresponding to X^a being a null vector field.

Corollary II.5. *Let X be a null infinitesimal symmetry of Δ . Then*

- $d\kappa^{(X)} = 0$
- if $\kappa^{(X)} \neq 0$, then there is a function $v : \Delta \rightarrow \mathbb{R}$ such that $dv \neq 0$ at every point of Δ , and

$$X^a D_a v = \kappa^{(X)} v \quad (2.32)$$

- if $\kappa^{(X)} = 0$, then there is a function $v : \Delta \rightarrow \mathbb{R}$ such that

$$X^b D_b (X^a D_a v) = 0 \quad (2.33)$$

on Δ , and

$$d(X^a D_a v) \neq 0 \quad (2.34)$$

at each point such that $(X^a D_a v) = 0$.

That means that an infinitesimal null symmetry of non-zero surface gravity can vanish only on a 2-dimensional slice of Δ , whereas in the case of the zero surface gravity it may vanish along a finite set of the geodesics. The last item is proved in [15].

Furthermore we distinguish the *cyclic* and *helical* infinitesimal symmetries, whose definitions are longer, therefore we spell them out more carefully.

Definition II.6. *Given a NEH (Δ, q_{ab}, D_a) , a vector field $\Phi^a \in \Gamma(T(\Delta))$ is cyclic infinitesimal symmetry whenever the following holds:*

- Φ^a is an infinitesimal symmetry of Δ (satisfies the equations (2.29)),
- the symmetry group of the maximal analytic extension $\bar{\Delta}$ it generates is diffeomorphic to $SO(2)$,
- Φ^a is spacelike at the points it doesn't vanish.

Definition II.7. *An infinitesimal symmetry X^a of a NEH (Δ, q_{ab}, D_a) is called helical if*

- The symmetry group generated by the projection \hat{X}^A of X^a onto the base space $\hat{\Delta}$ is diffeomorphic to $SO(2)$,
- in the maximal analytic extension $\bar{\Delta}$ there exists an orbit of the symmetry group generated by the extension \bar{X}^a which is not closed (i.e. it is diffeomorphic to a line).

A NEH admitting a helical infinitesimal symmetry will be called *helical*.

An important property of the latter is that by the local rigidity theorem it induces on Δ also a null and cyclic symmetry, that is

Theorem II.8. *Suppose the energy-momentum tensor T_{ab} satisfies the condition (2.20) for a non-vanishing null vector field ℓ^a tangent to a non-expanding horizon Δ . If Δ admits a helical infinitesimal symmetry X^a , then it also admits a cyclic infinitesimal symmetry Φ^a and a null infinitesimal symmetry ℓ^a such that*

$$X^a = \Phi^a + \ell^a. \quad (2.35)$$

Any cyclic symmetry admits in particular the choice of Δ foliation preserved by it, that is

Corollary II.9. *Suppose a non-expanding horizon Δ admits a cyclic infinitesimal symmetry Φ^a . Then there exists $\ell \in \Gamma(T(\Delta))$ such that*

$$\kappa^{(\ell)} = \text{const}, \quad [\Phi, \ell] = 0. \quad (2.36)$$

Moreover, for the maximal analytic extension $\bar{\Delta}$ of Δ there exists a diffeomorphism

$$h : \bar{\Delta} \rightarrow \hat{\Delta} \times \mathbb{R} \quad (2.37)$$

such that

$$h_* \Phi = (\hat{\Phi}, 0), \quad h_* \ell = (0, \partial_v), \quad (2.38)$$

where $\hat{\Phi} = \pi_* \Phi$ and v is the coordinate on \mathbb{R} .

2. The symmetries and invariants

An element new with respect to the situations studied in [15] is the presence and uniqueness of the invariants of NEHs introduced in section II A 4. It leads to the following results:

Theorem II.10. *Suppose that (Δ, q_{ab}, D_a) is an invariant-generic NEH (see Definition II.4) and X is its infinitesimal symmetry. Let ℓ^a and n_a are, respectively, the invariant vector field and the invariant covector field. Then*

a) $[X, \ell] = 0$, and $\mathcal{L}_X n_a = 0$.

b) *there exists a constant $a \in \mathbb{R}$ and a Killing vector field \hat{X}^A of the metric tensor \hat{q}_{AB} induced on the horizon base space $\hat{\Delta}$ such that*

$$X^a = a\ell^a + \tilde{X}^a \quad (2.39)$$

where \tilde{X}^a is tangent to the leafs of the invariant foliation of Δ , and

$$(\Pi_* \tilde{X})^A = \hat{X}^A. \quad (2.40)$$

III. SPACETIME NEIGHBORHOODS OF NEHS: INDUCED STRUCTURES

In the previous section we have defined invariant structures of generic, non-expanding horizons: the invariant vector ℓ^a , the invariant foliation and the invariant covector n_a . This intricate structure provides a natural extension to the spacetime neighbourhood of the horizon, defining in particular the coordinate system analogous to the Bondi one defined near null SCRI. This construction has been presented in [27] (and subsequently in [28]) in context of horizons in 4-dimensional spacetime and in [26] in general dimension. Here (Sec. III A) we perform the 1st step of this construction, extending the horizon invariants to the analogous invariant structures of its neighborhood. This extension will be next used in Sec. III C to define a natural way of describing the (possibly present) Killing fields on the neighbourhood of Δ .

A. The Bondi-like extension of a structure of a NEH.

The construction of the coordinate system is graphically presented in fig. III A. The detailed specification of the construction is as follows. Given a non-expanding horizon Δ let us fix a null, nowhere vanishing vector field $\ell^a \in \Gamma(T(\Delta))$ (not necessarily the invariant one), a foliation transversal to ℓ^a and preserved by its flow, and the corresponding covector n_a orthogonal to the foliation and normalized by the condition

$$\ell^a n_a = -1 \quad (3.1)$$

as in the previous section. The covector determines uniquely at Δ a null vector field \mathbf{n}^μ , tangent to the spacetime \mathcal{M} , such that the pullback of \mathbf{n}_μ onto Δ equals n_a . At each point of Δ , the vector \mathbf{n}^μ is transversal. We extend the vector field \mathbf{n}^μ to a null vector field defined in some spacetime neighbourhood of the horizon by the parallel transport along the null geodesics tangent to \mathbf{n}^μ at Δ . In other words we extend the vector field \mathbf{n}^μ from Δ to a vector field defined in a neighborhood of Δ such that

$$\nabla_{\mathbf{n}} \mathbf{n} = 0. \quad (3.2)$$

Assuming the vector field ℓ^a at the horizon is future (past) oriented the vector \mathbf{n}^μ is also future (past) oriented. Due to the finiteness of the transversal expansion of the vector field \mathbf{n}^μ at the horizon, there exists some region $\mathcal{M}' : \mathcal{M} \supset \mathcal{M}' \supset \Delta$ of the spacetime such that the geodesics generated by \mathbf{n}^μ define the foliation of \mathcal{M}' .⁵ We will denote the maximal (for given foliation of Δ) set of that property as the *domain of transversal null foliation*. In this region the field \mathbf{n}^μ is defined uniquely.

Via the flow of this vector field, the vector field ℓ^a defined on the horizon is extended to a vector field ζ^μ defined on \mathcal{M}' such that

$$\mathcal{L}_{\mathbf{n}} \zeta^\mu = 0, \quad \zeta^\mu|_{\Delta} = \ell^\mu. \quad (3.3)$$

⁵ The range of a region \mathcal{M}' strongly depends on the choice of the foliation the field \mathbf{n}^μ is orthogonal to.

We will refer to this field as to the *Bondi-like extension* of the vector field ℓ .

Generically, away from the horizon the vector field ζ^μ is no longer null. Indeed, the Lie derivative of $\zeta^\mu \zeta_\mu$ along \mathbf{n}^μ is at the horizon equal to

$$\mathcal{L}_{\mathbf{n}} \zeta^\mu \zeta_\mu|_\Delta = 2\kappa^{(\ell)} . \quad (3.4)$$

If $\kappa^{(\ell)} > 0$ (< 0) the vector ζ^μ becomes timelike near Δ , on the side from (into) which the geodesics defined by the vector \mathbf{n}^μ are incoming (outgoing). Hence, it can be treated as a 'time evolution' vector co-rotating with the horizon.⁶

Finally, the foliation of the horizon Δ is mapped by the flow of \mathbf{n}^μ into a foliation with $n-2$ surfaces diffeomorphic to the corresponding slices of Δ . The resulting foliation of the spacetime neighborhood of Δ is preserved by the flow of the vector field ζ^μ as well.

Also, there exists a uniquely defined function r in the neighborhood of Δ , such that

$$\mathbf{n}^\mu r_{,\mu} = -1 , \quad r|_\Delta = 0 , \quad (3.5)$$

called 'radial' coordinate on \mathcal{M}' . Given a value of r the flow of \mathbf{n} maps the horizon Δ into a cylinder $\Delta_{(r)}$. The radial coordinate r provides affine parametrization of the null geodesics tangent to \mathbf{n} . Each cylinder $\Delta_{(r)}$ is formed by the integral curves of the vector field ζ .

A parametrization of the integral curves of ζ can be fixed uniquely up to a constant, as a function v determined by its restriction to Δ and by

$$\zeta^\mu v_{,\mu} = 1 , \quad \mathbf{n}^\mu v_{,\mu} = 0 . \quad (3.6)$$

We are assuming that v is constant on the leaves of the foliation fixed on Δ . The condition $v = v_o$ defines an $n-1$ dimensional surface \mathcal{N}_{v_o} in the neighborhood of Δ consisting of the null geodesics tangent to \mathbf{n}^μ .

Remaining $n-2$ coordinates (x^A) can be defined in the neighborhood of Δ as the extension of any properly defined coordinate system \hat{x}^A given on the base space $\hat{\Delta}$ of the horizon

$$\forall_{p \in \Delta} \quad x^A(p) := \hat{x}^A(\pi^* p) , \quad \zeta^\mu x^A_{,\mu} = \mathbf{n}^\mu x^A_{,\mu} = 0 , \quad (3.7)$$

where π is a projection onto base space defined via (2.7).

The set of n functions (x^A, v, r) defined above forms at \mathcal{M}' a well defined coordinate system. The coordinates v and r are defined globally on \mathcal{M}' . On the other hand, the coordinates \hat{x}^A are defined locally on elements of an open covering of $\hat{\Delta}$, next pulled back to Δ and finally extended to \mathcal{M}' . Due to the similarity with the coordinate system defined by Bondi at the null SCRI it will be referred to as the *Bondi-like coordinate system* of the horizon spacetime neighbourhood. In these coordinates

$$\zeta^\mu = (\partial_v)^\mu , \quad \mathbf{n}^\mu = -(\partial_r)^\mu , \quad \mathbf{n}_\mu = -(\mathrm{d}v)_\mu . \quad (3.8)$$

B. Invariants of NEHs neighborhoods

We use the Bondi extension to endow the neighborhood of an invariant-generic NEH Δ with invariant structures. Let the starting point for the construction of the previous subsection be the invariant vector field ℓ^a of the geometry of Δ , the invariant foliation and the invariant covector n_a . Then we get the following structures invariantly defined in the neighborhood of Δ : the vector fields ζ^μ , and \mathbf{n}^μ , the foliation by the surfaces $\Delta_{(r)}$, the foliation by the surfaces \mathcal{N}_v .

Definition III.1. *We will call ζ^μ the invariant vector field, \mathbf{n}_μ the invariant covector field, and the foliations, the invariant foliations, respectively of the neighborhood of Δ . And we will refer to the coordinates (x^A, v, r) as to the Bondi invariant coordinates.*

⁶ Exterior/interior of Δ is undefined at this point. One can define exterior to be that side of Δ at which near to Δ the vector field is timelike.

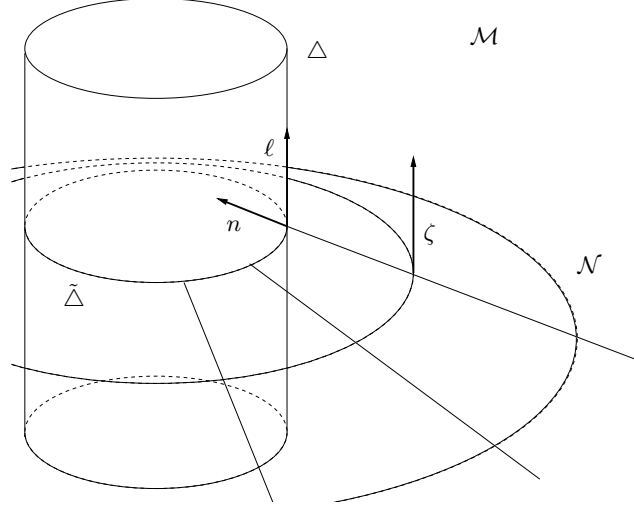


FIG. 1. The construction of Bondi-like coordinate system.

C. The invariants and the Killing vectors

The Bondi invariant coordinate system introduced in the previous subsection in the neighborhood of Δ is very well suited to identify and characterize the Killing vectors possibly existing in the neighborhood of Δ .

Lemma III.2. *Consider a non-expanding horizon Δ . Assume it is invariant-generic. Let ζ^μ and \mathbf{n}_μ be the invariant, respectively vector field and covector field of a neighborhood of Δ . Suppose K^μ is a Killing vector field defined in the spacetime \mathcal{M} and tangent to Δ . Then,*

1. *the restriction of K^μ to Δ is an infinitesimal symmetry of the geometry of Δ , and*
2. $[K, \zeta] = [K, \mathbf{n}] = 0.$ (3.9)

The first item is obvious and the second follows from the fact that there is an isometric flow of K^μ defined in a spacetime neighborhood of any cross-section of $\pi : \Delta \rightarrow \hat{\Delta}$.

The infinitesimal symmetries were characterized in terms of the invariant vector ℓ^a and the invariant foliation in the previous section. Owing to Lemma III.2 we can characterize the Killing vectors by the invariant vector field ζ^μ and the invariant foliations, that is by the Bondi invariant coordinates.

Theorem III.3. *Suppose the assumptions of Lemma (III.2) are satisfied. Then*

- (i) *if the restriction of K^μ to Δ is null everywhere on Δ , then there is a constant $a_0 \in \mathbb{R}$ such that*

$$K^\mu = a_0 \zeta^\mu; \quad (3.10)$$

- (ii) *If K^μ is not null at Δ , then it takes the following form in the Bondi-like invariant coordinate system system (x^A, r, v)*

$$K^\mu = a_0 \zeta^\mu + [\Phi^A(x^B) \partial_A]^\mu, \quad (3.11)$$

where

$$k^a = a_0 \ell^a + [\Phi^A(\hat{x}^B) \partial_{\hat{x}^A}]^a, \quad a_0 \in \mathbb{R} \quad (3.12)$$

is an infinitesimal symmetry of Δ .

Proof. The conclusion follows from the following calculation

$$0 = [n, K] = -[\partial_r, K^\mu \partial_\mu] = -\partial_r(K^\mu) \partial_\mu \quad (3.13)$$

of which a direct consequence is

$$K^\mu(x^A, v, r) = K^\mu(x^A, v, 0) = k^\mu(x^A, v) . \quad (3.14)$$

□

To see the strength of this result, let us note that, given the Bondi invariant coordinates (x^A, v, r) , and the Killing vectors $\hat{k}_1^A(\hat{x}^B)\partial_{\hat{x}^A}, \dots, \hat{k}_m^A(\hat{x}^B)\partial_{\hat{x}^A}$ of the horizon base space $\hat{\Delta}$ geometry \hat{q}_{AB} , if we want to know if there is a spacetime Killing vector field tangent to the horizon, all we have to do is to check candidates K^μ of the form

$$K = a_0\partial_v + (a^1\hat{k}_1^A + \dots + a^m\hat{k}_m^A)\partial_{x^A} \quad (3.15)$$

for all $a_0, \dots, a_m \in \mathbb{R}$.

D. The Killing vectors in non-invariant Bondi coordinates

Even if we are not assuming that a NEH Δ is invariant-generic, a Killing vector field still takes a simple form in suitably chosen Bondi-like coordinates.

Theorem III.4. *Consider a non-expanding horizon (Δ, q_{ab}, D_a) contained in a spacetime (\mathcal{M}, g) ; suppose (\mathcal{M}, g) admits a Killing field K^μ tangent to Δ ; suppose there is a nowhere vanishing vector field $\ell^\mu \in \Gamma(T(\Delta))$ such that the restriction k^a of K^μ to Δ satisfies*

$$[\ell, k] = 0 \quad (3.16)$$

and a foliation of Δ by sections of $\pi : \Delta \rightarrow \hat{\Delta}$ preserved by the flow of k^a . Then, there are coordinates (\hat{x}^A, v) on Δ such that

$$k = a\partial_v + b\Phi^A(x^B)\partial_A , \quad a, b \in \mathbb{R} , \quad (3.17)$$

and in the corresponding Bondi like coordinates (x^A, r, v) the Killing field K^μ is necessarily of the form:

$$K = a\partial_v + b\Phi^A(x^B)\partial_A , \quad a, b = \text{const} . \quad (3.18)$$

IV. 4D EINSTEIN-MAXWELL NEIGHBOURHOODS OF 3D NEHS

In the remaining part of the article we restrict our studies to the spacetime neighbourhood of the horizon being of dimension 4. The focus point of this section is the analysis of the spacetime metric expansion in the radial (the vector field \mathbf{n}^μ) direction first without restrictions on the spacetime matter content and next assuming that the only matter admitted is the Maxwell field. In the latter case we discuss the characteristic Cauchy problem on Δ .

We start with a short summary of the structure and evolution equations of the Maxwell field on a NEH. Next we provide a short comprehensive discussion of the Cauchy problem on electro-vacuum NEH.

Although the description of an electrovac neighbourhood of a NEH and the initial value problem corresponding to it can be (and are) formulated in the geometric formalism used so far in the article, it is particularly convenient (especially in the task of providing the metric expansion) to perform the analysis using appropriately chosen Newman-Penrose null frame (see Appendix A). Therefore, we introduce such frame (in subsection IV C) adopted to the Bondi-like coordinate system defined in Section III further providing the dictionary between the formerly used geometric quantities and the new objects defined by the frame. This construction is then used to analyze the radial expansion of the spacetime metric and determine the initial data (at the horizon) necessary to uniquely specify the terms of expansion for any matter content and next, with restriction to electrovac spacetime.

Next, in subsection IV D we restrict our interest solely to electrovac horizon neighbourhoods. To do so we reintroduce the dictionary between the two formalisms (geometric and the null frame one) provided earlier, this time adopting it to context at hand. Using it we discuss the properties of Maxwell field equations in the horizon neighbourhood. These properties are next used to reexamine the set of initial data needed to specify the expansion of metric previously studied in general matter case.

The section is concluded by the detailed discussion of the characteristic initial value problem for electrovac case with initial surfaces defined by Bondi-like coordinate extension (subsection IV D 2) and with use of the null frame introduced in Section IV C. In all the subsections for the reader's convenience we just present the results, showing the calculations in the appendices B and C.

A. Electromagnetic field on a NEH.

Suppose the spacetime \mathcal{M} is 4 dimensional, and an electromagnetic field $F_{\mu\nu}$ is present in a neighbourhood of Δ . We will be considering the Einstein-Maxwell equations in detail in section IV D in terms of the Newman-Penrose components. But before that, let us recall the equivalent geometric description of the constraints imposed on the components of $F_{\alpha\beta}$ by the Maxwell vacuum equations [17].

As in the earlier parts of this article, given a spacetime covector W_α , its pullback on Δ is denoted by W_a . Similarly for any covector W_a orthogonal at $p \in \Delta$ to ℓ^a , we denote the corresponding element of $(T_p\Delta/L)^*$ by W_A .

The electromagnetic field energy-momentum tensor $T_{\alpha\beta}$ satisfies Stronger Energy Condition II.1, hence T_{ab} satisfies (2.20). The tensor T_{ab} contributes to the constraints (2.19), and the condition (2.20) amounts to the vanishing of some components of $F_{\alpha\beta}$ at Δ , namely

$$\ell^a (F - i \star_{\mathcal{M}} F)_{ab} = 0, \quad (4.1)$$

where $\star_{\mathcal{M}}$ is the spacetime Hodge star.

We assume that $F_{\mu\nu}$ satisfies in a spacetime neighbourhood of Δ the Maxwell vacuum equations

$$d(F - i \star_{\mathcal{M}} F) = 0. \quad (4.2)$$

The Maxwell equations constrain further the remaining components of $F_{\mu\nu}$ at Δ . In the section IV C 1 we derive these constraints by using a null frame adapted to Δ . Here we express the result in a frame independent way.

The first of Maxwell constraints reads

$$\ell^a \star_{\mathcal{M}} d(F - i \star_{\mathcal{M}} F)_a = 0. \quad (4.3)$$

Therefore the covector $\star_{\mathcal{M}} d(F - i \star_{\mathcal{M}} F)_a$ is orthogonal to ℓ and can be subject to the horizon Hodge dualization \star_{Δ} introduced in Section II A 1. The second constraint following from the Maxwell equations takes the form of the horizon self-duality condition

$$\star_{\Delta} (\star_{\mathcal{M}} d(F - i \star_{\mathcal{M}} F))_A = i \star_{\mathcal{M}} d(F - i \star_{\mathcal{M}} F)_A. \quad (4.4)$$

B. Geometry of the Cauchy problem on Δ

As a null surface, a NEH Δ is not a part of a typical surface to formulate the initial value problem for the Einstein-matter field equations – a Cauchy problem for general relativity. A suitable formulation is known as the characteristic Cauchy problem [29, 30]. We will apply it to the NEHs in this subsection, using the null frame approach to gravity called the Newman-Penrose framework. Here, we express the outline of the results in a geometric, frame independent way. The details are presented in further subsections: IV C through IV D and in Appendix C.

Consider a NEH (Δ, q_{ab}, D_a) in a spacetime $(\mathcal{M}, g_{\alpha\beta})$. The Bondi-like coordinate systems (introduced in section 3.1) adapted to Δ form an atlas on the neighborhood of Δ . Let (x^A, v, r) be one of these coordinate systems.

At any point of Δ , the spacetime metric tensor can be written in the form

$$g_{\alpha\beta} dx^\alpha \otimes dx^\beta = q_{AB} dx^A \otimes dx^B - dv \otimes dr - dr \otimes dv. \quad (4.5)$$

Furthermore, the derivative $\partial_r g_{\alpha\beta}$ for all $\alpha, \beta = 1, 2, 3, 4$ is determined at Δ by the components of the connection D_a (see (4.32) below). The higher derivatives $\partial_r^k g_{\alpha\beta}$ are determined by the Einstein-matter field equations and data defined on a 2-dimensional slice $\tilde{\Delta}$ of Δ [29].

1. The Einstein vacuum case: pure gravitational field

Suppose, in a neighbourhood of Δ the vacuum Einstein equations hold. Then, the horizon geometry q_{ab}, D_a satisfies the constraints (2.19) with arbitrary null vector field $\ell^a = (\partial_v)^a$ and

$$\mathcal{R}_{ab}^{(4)} = 0. \quad (4.6)$$

To determine the derivatives $(\mathcal{L}_\ell)^k g_{\alpha\beta}$, $k = 2, \dots, n$ at Δ it is sufficient to know at a slice of Δ ,

$$\tilde{\Delta}_{v_0} = \{x \in \Delta : v(x) = v_0\} \quad (4.7)$$

(assuming it is a global section of $\pi : \Delta \rightarrow \hat{\Delta}$), the shear (see 4.14c below) λ of the (transversal to Δ , orthogonal to the slice, null vector field) $n^\mu = (\partial_r)^\mu$, and all its derivatives

$$\partial_r^k \lambda, \quad k = 0, \dots, n-2. \quad (4.8)$$

The above statement can be demonstrated explicitly (“exactly”) by solving the hierarchy of the ordinary differential equations (ODE’s) - see the next subsections and Appendix B 1.⁷

In order to determine the spacetime metric $g_{\alpha\beta}$ in some 4-dimensional region containing Δ , we can use the 3-dimensional surface in \mathcal{M} : $\mathcal{N}_{v_0} = \{x \in \mathcal{M} : v(x) = v_0\}$ spanned by the null geodesics tangent to the vector field n^μ and intersecting the slice $\tilde{\Delta}_{v_0}$ of Δ specified in (4.7). Then, the following data:

- 1) on Δ : q_{ab}, D_a such that (2.19) with $\mathcal{R}_{ab}^{(4)} = 0$
- 2) on \mathcal{N}_{v_0} : λ with certain boundary condition specified on $\lambda|_{\Delta \cap \mathcal{N}_{v_0}}$ induced by D_a (see (2.19))

determine the spacetime metric tensor (modulo diffeomorphisms) in the domain of dependence of $\Delta \cup \tilde{\Delta}_{v_0}$. Moreover, as we vary the vacuum spacetime metric tensor in such a way that Δ is a NEH, and \mathcal{N}_{v_0} satisfies the definition, the data 1)-2) above ranges all the possible NEH geometries on Δ such that (2.19, 4.6), and all possible functions $\lambda : \mathcal{N}_{v_0} \rightarrow \mathbb{C}$ which satisfy the boundary condition at $\Delta \cap \mathcal{N}_{v_0}$.

2. The Einstein-Maxwell vacuum case: pure gravitational and Maxwell field

Suppose now, in the neighbourhood \mathcal{M}' the vacuum Einstein-Maxwell equations hold. Then, the horizon geometry (q_{ab}, D_a) and the pullbacks $F_{ab}, \star_{\mathcal{M}} F_{ab}$ satisfy the constraints (4.1, 4.3, 4.4). At every point of Δ , all the transversal derivatives $(\mathcal{L}_\ell)^k g_{\alpha\beta}$ as well as $(\mathcal{L}_\ell)^k F_{\alpha\beta}$ can be determined by certain components of the horizon geometry and electromagnetic field on it (see the next subsections and Appendix C 1). More precisely, the initial value problem is again the characteristic Cauchy problem with the initial data null surfaces $\Delta, \mathcal{N}_{v_0}$ specified exactly as in previous sub-subsection. Then in order to determine the spacetime metric $g_{\alpha\beta}$ and the electromagnetic field $F_{\alpha\beta}$ in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$, it is sufficient to specify the following data:

- 1) on Δ : q_{ab}, D_a, F_{ab} and $\star_{\mathcal{M}} F_{ab}$ such that (4.1, 4.3, 4.4) hold,
- 2) on \mathcal{N}_{v_0} : λ with the boundary data $\lambda|_{\Delta \cap \mathcal{N}_{v_0}}$ induced by D_a and the pullbacks $n^\mu F_{\mu a}$ and $n^\mu \star_{\mathcal{M}} F_{\mu a}$ satisfying the boundary conditions induced by the Maxwell equations (4.2).

Moreover, as we vary the vacuum solutions $g_{\mu\nu}$ and $F_{\mu\nu}$ to the Einstein-Maxwell equations such that Δ is a NEH and \mathcal{N}_{v_0} satisfies the definition, the data 1)-2) above ranges: 1) all the NEH geometries and the electromagnetic fields on Δ such that (4.1, 4.3, 4.4) and, 2) all the possible functions $\lambda : \mathcal{N}_{v_0} \rightarrow \mathbb{C}$ which satisfy the appropriate boundary condition at $\Delta \cap \mathcal{N}_0$ listed in point 1) and one-forms $n^\mu F_{\mu a}$ and $n^\mu \star_{\mathcal{M}} F_{\mu a}$ defined on \mathcal{N}_{v_0} also satisfying the appropriate boundary condition at $\Delta \cap \mathcal{N}_0$ listed in point 2).

Below we present the expansion, specification of the data and the boundary conditions in detail using the Newman-Penrose framework.

C. The null frame, the metric expansion

Our starting point is the Bondi-like extension of the structures and coordinates defined in Section III A on a NEH Δ : a null, nowhere vanishing vector field ℓ^a tangent to Δ , a function $v : \Delta \rightarrow \mathbb{R}$ such that $\ell^a D_a v = 1$, and coordinates (x^A, v) on Δ such that $\ell^a D_a x^A = 0$. Whereas v is globally defined, the coordinates x^A are defined locally, form an atlas, the pullback by π^* of atlas \hat{x}^A defined on the base manifold $\hat{\Delta}$. The function v defines on Δ the covector $n = -dv$, which in the neighborhood \mathcal{M}' of Δ , defines in particular the null vector field n^μ . Using this structure we define below a null frame (see Appendix A for the basic properties of null frames) $(e_1, e_2, e_3, e_4) = (m, \bar{m}, n, \ell)$ such that

$$(e_3)^\mu = n^\mu \text{ in } \mathcal{M}', \quad (e_4)^\mu = \ell^\mu \text{ at } \Delta, \quad (4.9)$$

⁷ Actually, the initial data the geometry (q_{ab}, D_a) defines on the slice is also sufficient to determine (q_{ab}, D_a) on the whole Δ).

and e_1, e_2 are tangent to the leaves of the foliation of Δ . Whereas the vector fields e_3 and e_4 are defined on the entire neighborhood \mathcal{M}' , the domains of the vector fields e_1 and e_2 will coincide with those of the coordinates (x^A) . By (e^1, e^2, e^3, e^4) we will be denoting the dual coframe. This construction of the frame has been already discussed in [26, 27] and subsequently presented in [28].

The spacetime metric tensor $g_{\mu\nu}$ on \mathcal{M}' and the degenerate metric tensor q_{ab} induced on Δ take in that frame the following form:

$$g_{\mu\nu} = (e^1 \otimes e^2 + e^2 \otimes e^1 - e^3 \otimes e^4 - e^4 \otimes e^3)_{\mu\nu} , \quad (4.10a)$$

$$q_{ab} := g_{ab} = (e^1 \otimes e^2 + e^2 \otimes e^1)_{ab} . \quad (4.10b)$$

where $(\cdot)_{ab}$ stands for the pull-back to Δ of a tensor originally defined onto \mathcal{M} .

1. Geometry and constraints at the horizon, the invariants

Let us now focus on the properties of the frame at Δ itself. For this purpose through this sub-subsection we will adopt the shortened notation, using ‘=’ for ‘=| Δ ’. In the Bondi-like coordinates (x^A, v, r) defined in Section III A, $r = 0$ on Δ , and at Δ the vector field ∂_v is null

$$\ell^a = (\partial_v)^a \quad (4.11)$$

and has constant surface gravity. The real vectors $\Re(m)^\mu, \Im(m)^\mu$ are (automatically) tangent to Δ . To adapt the frame further, we assume the vector fields $\Re(m)^a, \Im(m)^a$ tangent to the constancy surfaces $\tilde{\Delta}_v$ of the coordinate v (3.6) are Lie dragged by the flow $[\ell]$

$$\mathcal{L}_\ell m^a = 0 . \quad (4.12)$$

This implies immediately, that the projection of m^a onto $\hat{\Delta}$ uniquely defines on a horizon base space $\hat{\Delta}$ a null vector frame $(\hat{m}, \bar{\hat{m}})$ and the differential operators $\delta, \bar{\delta}$

$$(\pi_* m)^A =: \hat{m}^A , \quad \delta := \hat{m}^A (x^B) \partial_A \quad (4.13)$$

corresponding to the frame vectors.

The frame specified above is adapted to: the vector field ℓ^a , the flow of ℓ^a invariant foliation of Δ , and the null complex-valued frame \hat{m}^A defined on the manifold $\hat{\Delta}$. Spacetime frames constructed in this way on Δ will be called *adapted*.

Since all the frame elements are Lie dragged by ℓ^a , the connection D induced on Δ can be decomposed as follows⁸

$$m^\nu D \bar{m}_\nu = \pi^* \left(\hat{m}^A \hat{D} \bar{\hat{m}}_A \right) =: \pi^* \hat{\Gamma} , \quad (4.14a)$$

$$-n_\nu D \ell^\nu = \omega^{(\ell)} = \pi e_{(\Delta)}^2 + \bar{\pi} e_{(\Delta)}^1 + \kappa^{(\ell)} e_{(\Delta)}^3 , \quad (4.14b)$$

$$-\bar{m}^\nu D n_\nu = \mu e_{(\Delta)}^1 + \lambda e_{(\Delta)}^2 + \pi e_{(\Delta)}^4 , \quad (4.14c)$$

$$m_\mu D \ell^\mu = 0 , \quad (4.14d)$$

where $\hat{\Gamma}$ is the Levi-Civita connection 1-form corresponding to the covariant derivative \hat{D} defined by \hat{q} and to the null frame \hat{m}^A defined on $\hat{\Delta}$

$$\hat{\Gamma} =: 2\bar{a}\hat{e}^1 + 2a\hat{e}^2 . \quad (4.15)$$

The rotation 1-form potential $\omega^{(\ell)}$ in the chosen frame takes the form

$$\omega^{(\ell)} = \pi e_{(\Delta)}^2 + \bar{\pi} e_{(\Delta)}^1 - \kappa^{(\ell)} e_{(\Delta)}^4 , \quad (4.16)$$

In terms of the coordinates (x^A, v) on Δ , the functions a and π satisfy

$$\partial_v a = \partial_v \pi = 0 . \quad (4.17)$$

⁸ The decomposition is consistent with the definition of connection coefficients presented in A.

The Ricci tensor is represented by the set of the Newman-Penrose coefficients Φ_{ij} , $i, j = 0, 1, 2$ (A7). In terms of them, the constraints induced on the horizon geometry (q_{ab, D_a}) by the Einstein field equations described in section II A 3 are by the identity (2.19) equivalent to the following set of equations

$$8\pi G(T_{m\bar{m}} - \frac{1}{2}T_{q_{m\bar{m}}}) = -2(\Phi_{11} + 3 \overset{(4)}{\mathcal{R}}) = 2D\mu + 2\kappa^{(\ell)}\mu - \tilde{\text{div}}\tilde{\omega}^{(\ell)} - |\tilde{\omega}^{(\ell)}|_{\tilde{q}}^2 + \overset{(2)}{\mathcal{R}}_{m\bar{m}} , \quad (4.18a)$$

$$8\pi GT_{\bar{m}\bar{m}} = -2\Phi_{20} = 2D\lambda + 2\kappa^{(\ell)}\lambda - 2\bar{\delta}\pi - 4a\pi - 2\pi^2 , \quad (4.18b)$$

where $D := \ell^a \partial_a$, $\delta := m^a \partial_a$, $\overset{(2)}{\mathcal{R}}_{m\bar{m}} := (\pi^* \overset{(2)}{\mathcal{R}})_{ab} m^a \bar{m}^b$ and $(\tilde{\text{div}}\tilde{\omega}^{(\ell)})$ is the divergence of projected rotation 1-form (2.12). As functions of the variables x^A (3.7), by (2.5) and (2.15) the latter two objects equal their counterparts $(\overset{(2)}{\mathcal{R}}_{m\bar{m}}, \hat{\text{div}}\hat{\omega}^{(\ell)})$ defined on the horizon base space

$$\overset{(2)}{\mathcal{R}}_{m\bar{m}} := 2\delta a + 2\bar{\delta}\bar{a} - 8a\bar{a} , \quad (4.19a)$$

$$\hat{\text{div}}\hat{\omega}^{(\ell)} = \delta\pi + \bar{\delta}\bar{\pi} - 2a\bar{\pi} - 2\bar{a}\pi . \quad (4.19b)$$

Remark IV.1. *In terms of the Newman-Penrose coefficients, Definition II.3 of the natural vector field ℓ^a of a NEH geometry reads: ℓ^a is tangent to Δ , null, $\kappa^{(\ell)} = 1$ and*

$$\ell^a \mu_{,a} = 0 . \quad (4.20)$$

The invariant foliation listed in Definition II.4 and the corresponding invariant variable v is defined by the following condition

$$\delta\pi + \bar{\delta}\bar{\pi} - 2a\bar{\pi} - 2\bar{a}\pi = 0 . \quad (4.21)$$

Furthermore, the condition (2.21) reads

$$D\Phi_{20} = D(\Phi_{11} + 3 \overset{(4)}{\mathcal{R}}) = 0 , \quad (4.22)$$

whereas the functions Φ_{11} , Φ_{20} and $\overset{(4)}{\mathcal{R}}$ are determined by the electromagnetic field (which we will see below).

Some components the energy-momentum and Weyl tensor vanish due to the Stronger Energy Condition II.1. Indeed, the following Ricci tensor components (listed in (4.23a)) vanish on Δ due to (2.6), and the Weyl tensor components (listed in (4.23b)) due to the definition of NEH and the Bianchi equalities (see (A7) for the definition of components)

$$\Phi_{00} = \Phi_{01} = \Phi_{10} = 0 . \quad (4.23a)$$

$$\Psi_0 = \Psi_1 = 0 , \quad (4.23b)$$

Moreover, the horizon geometry and the matter fields at Δ determine the values at the horizon of the Weyl tensor components Ψ_2 and Ψ_3 (via the NP equations (A8b) and (A8c) respectively),

$$\Psi_2 = -\frac{1}{4} \overset{(2)}{\mathcal{R}} - \frac{1}{2}(\delta\pi - \bar{\delta}\bar{\pi}) - a\bar{\pi} + \bar{a}\pi + \Phi_{11} + \frac{1}{24} \overset{(4)}{\mathcal{R}} \quad (4.24a)$$

$$\Psi_3 = \bar{\delta}\mu - \delta\lambda + \pi\mu + (4\bar{a} - \bar{\pi})\lambda + \Phi_{12} , \quad (4.24b)$$

The remaining one: Ψ_4 , is constrained by Bianchi identity (A11d) which at the horizon reads

$$\begin{aligned} D\Psi_4 &= -2\kappa^{(\ell)}\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 \\ &\quad - \bar{\mu}\Phi_{20} + (\pi + 2a)\Phi_{21} - 2\lambda\Phi_{11} - \Phi_{20,r} . \end{aligned} \quad (4.25)$$

In this way, Ψ_4 at Δ is uniquely determined by the value of Ψ_4 on chosen section, the horizon geometry and the matter fields.

2. Extension to the spacetime neighbourhood

Given the coframe (e^1, \dots, e^4) dual to the frame (m, \bar{m}, n, ℓ) defined above at Δ , the condition

$$\nabla_n e^\mu = 0 \quad (4.26)$$

defines its unique extension to the spacetime neighbourhood \mathcal{M}' . The corresponding connection coefficients $g(e_\gamma, \nabla_\alpha e_\beta)$ are defined in (A4). In the Bondi-like coordinate system this adapted (co)frame extended by (4.26) takes the form

$$e_1 = m = \bar{e}_2 = m^A(\partial_A + Z_A \partial_r), \quad e^1 = \bar{e}^2 = \bar{m}_A dx^A + X dv, \quad (4.27a)$$

$$e_3 = n = -\partial_r, \quad e^3 = -dr + Z_A dx^A + H dv, \quad (4.27b)$$

$$e_4 = \ell = \partial_v - \bar{X} e_1 - X e_2 + H \partial_r, \quad e^4 = dv, \quad (4.27c)$$

where Z_A, H are real functions, m^A, X are complex. At the horizon these functions take the following values

$$X|_\Delta = H|_\Delta = Z_A|_\Delta = 0 \quad m_A|_\Delta = \pi^* \hat{m}_A. \quad (4.28)$$

The condition (4.26) (consistent with $\nabla_n n^\mu = 0$ and $n^\mu n_\mu = \text{const}$) imposes on the connection coefficients (defined via (A4)) corresponding to it the following constraints true in \mathcal{M}'

$$\tau = \gamma = \nu = \mu - \bar{\mu} = \pi - (\alpha + \bar{\beta}) = 0. \quad (4.29)$$

In particular the last constraint allows us to express the coefficients (α, β) in terms of π and

$$a := \frac{1}{2}(\alpha - \bar{\beta}). \quad (4.30)$$

The commutators of the differential operators corresponding to the frame vectors can be expressed in terms of the functions (H, X, m^A, Z_A) and their derivatives. On the other hand they are determined by the connection coefficients via (A10). That correspondence leads to the constraints on the frame coefficients which determine their evolution of the functions (X, H, m_A, Z_A) along the transversal to Δ null geodesics:

$$-\partial_r X = \bar{\pi} + \mu X + \bar{\lambda} \bar{X} \quad (4.31a)$$

$$\partial_r H = (\epsilon + \bar{\epsilon}) + \pi X + \bar{\pi} \bar{X} \quad (4.31b)$$

$$\partial_r m_A = \bar{\lambda} \bar{m}_A + \mu m_A \quad (4.31c)$$

$$\partial_r Z_A = \pi m_A + \bar{\pi} \bar{m}_A \quad (4.31d)$$

This set is supplemented by analogous evolution equations for the spin (connection) coefficients (C2c-C2f, C3a, C3b, C4) and Weyl tensor components (C2h, C3d, C6, C5).

The global structure of the resulting frame is as follows: The neighborhood \mathcal{M}' of a given NEH Δ is covered by open sets \mathcal{U}_I , $I = 1, \dots, K$ obtained from a covering $\hat{\mathcal{U}}_I$, $I = 1, \dots, K$ of the base $\hat{\Delta}$. Each open set \mathcal{U}_I is the union of the null geodesics tangent to n^μ or to ℓ^μ , and intersecting the set $\hat{\mathcal{U}}_I$, for every $I = 1, \dots, K$.

3. Metric expansion at the horizon

Since in this sub-subsection we consider objects on Δ only, we again adopt the notation $\cdot' \equiv \cdot|'_\Delta$.

It is a straightforward observation that the horizon geometry (q_{ab}, D_a) already determines the frame components at Δ (through (4.28)) as well as their 1-order radial derivative ∂_r (via (4.31)),

$$X_{,r} = -\bar{\pi} \quad (4.32a)$$

$$H_{,r} = \kappa^{(\ell)} \quad (4.32b)$$

$$Z_{A,r} = \pi m_A + \bar{\pi} \bar{m}_A \quad (4.32c)$$

$$m_{A,r} = \bar{\lambda} \bar{m}_A + \mu m_A \quad (4.32d)$$

$$(4.32e)$$

The second order of the frame expansion, following directly from (A8j, A8n, A8m, A8l) is

$$X_{,rr} = -\bar{\Psi}_3 - \Phi_{12}, \quad (4.33a)$$

$$H_{,rr} = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (4.33b)$$

$$Z_{A,rr} = (\Psi_3 + \Phi_{21})m_A + (\bar{\Psi}_3 + \Phi_{12})\bar{m}_A \quad (4.33c)$$

$$m_{A,rr} = -\Phi_{22}m_A - \bar{\Psi}_4\bar{m}_A. \quad (4.33d)$$

Note that the derivatives $H_{,rr}, X_{,rr}, Z_{A,rr}$ on Δ are determined directly by (q_{ab}, D_a) and the Ricci tensor [see (4.24a, 4.24b, 4.25)]. The last derivative, $m_{A,rr}$, involves a solution Ψ_4 to the equation (4.25) uniquely determined by the initial value of Ψ_4 on chosen section and the horizon geometry.

To summarize, by direct inspection of the system of equations used here we see, that the data which is not determined, thus must be specified, consists of the following components:

- (i) $\Phi_{21}, \Phi_{22}, \overset{(4)}{\mathcal{R}}, \Phi_{20,r}$ given on the entire Δ , and
- (ii) Ψ_4 given on an initial slice $\tilde{\Delta}$.

Remark IV.2. *As it is pointed out at the end of the previous Section IV C 2, the elements frame e_1 and e_2 of the frame are defined locally, on the sets $\pi^{-1}(\tilde{U}_I)$, $I = 1, \dots, K$ covering the horizon Δ . On each intersection between two sets, say $\pi^{-1}\mathcal{U}_I$ and $\pi^{-1}\mathcal{U}_J$, there is an obvious transformation law,*

$$e_1^{(I)} = u(x^A)^{(IJ)} e_1^{(J)} \quad (4.34)$$

where $u(x^A)^{(IJ)} \in U(1)$. On the other hand, e_3 and e_4 are defined globally, at every point of Δ . Now, the functions Φ_{21}, Φ_{22} and Φ_{20} , as components of a tensor, are also defined locally, on each set $\pi^{-1}(\tilde{U}_I)$, and satisfy the corresponding transformation laws on the intersections. On the other hand, the Weyl tensor component Ψ_4 is not sensitive to the frame transformations preserving e_3 and e_4 , and $\overset{(4)}{\mathcal{R}}$ is just a scalar. The derivative $\partial_r = e_4^\mu \partial_\mu$, hence it is defined globally and commutes with the transformations.

Finally, we address the question, what data is required to determine the general order derivatives $\partial_r^n (e^\mu)_\nu$. It turns out, that the general case is described by the following:

Corollary IV.3. *Given a NEH Δ in a 4-dimensional spacetime satisfying the Einstein field equations with a general kind of matter, the Bondi-like coordinates (x^A, v, r) defined in Section III A and a null frame (e_1, e_2, e_3, e_4) defined in Section IV C 1 and IV C 2, the following data*

- (i) *the value of the constant $\kappa^{(\ell)}$,*
- (ii) *on the initial slice $\tilde{\Delta}$: m^α (which is tangent to the slice by the construction), π, μ, λ and $\partial_r^k \Psi_4 \forall k \in \{0, \dots, n-2\}$*
- (iii) *on Δ : $\partial_r^k \Phi_{11}, \partial_r^k \Phi_{21}, \partial_r^k \Phi_{22}, \partial_r^k \overset{(4)}{\mathcal{R}}, \partial_r^{k+1} \Phi_{20} \forall k \in \{0, \dots, n-2\}$*

determines uniquely all the radial derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ (at Δ) of the frame components up to the order $k = n$. The data is free, that is it is not subject to any extra constraints, modulo Remark IV.2. Also, in the current work we make the additional assumption (2.21) which in terms of the Newman-Penrose coefficients reads: $\partial_v \Phi_{02} = \partial_v \Phi_{20} = \partial_v (\Phi_{11} + 3 \overset{(4)}{\mathcal{R}}) = 0$.

For the detailed proof of the above corollary the reader is referred to Appendix B. At this point one has to remember though, that Corollary IV.3 is not an existence or a uniqueness statement. For that, the data on Δ has to be completed by suitable data defined on another null surface. Also the Einstein equations on $g_{\mu\nu}$ have to be completed by equations satisfied by the matter which contributes to the energy-momentum tensor (see Section IV B).

The above expansion has been discussed in [26] and subsequently presented up to a 2nd order (also specifically in Einstein-Maxwell case) in [28].

D. 4-dimensional electrovac NEH

Let us now restrict our studies to the case, when \mathcal{M}' admits electromagnetic field as a sole matter content. The geometry of a non-expanding horizon in that case was analyzed already in [7]. Here we extend these studies by analysis of the properties of an electrovac NEH's spacetime neighbourhood. First in Section IV D 1 we introduce the necessary geometric objects used for the description of the Maxwell fields, discuss their properties and describe how the Maxwell evolution equations influence the set of data necessary to determine the metric expansion at the horizon. Next in Section IV D 2 we discuss the characteristic initial value problem for the system under consideration in context of Bondi-like coordinate system introduced in Section III.

Given a NEH Δ of a geometry (q_{ab}, D_a) we use throughout this subsection the following objects:

- the Bondi-like coordinates (x^A, v, r) adapted to Δ and such that $\ell^a = (\partial_v)^a$ at Δ is a null vector of a constant surface gravity $\kappa^{(\ell)}$,
- the null tangent frame $(e_1, \dots, e_4) = (m^\mu, \bar{m}^\mu, n^\mu, \ell^\mu)$ of the form (4.27) and the dual coframe (e^1, \dots, e^4) .

1. Constraints and the metric expansion

Given a null frame specified above the electromagnetic field can be represented by the field coefficients defined in the following (equivalent to (A12)) way⁹

$$F := \frac{1}{2} F_{\mu\nu} e^\mu \wedge e^\nu = -\Phi_0 e^4 \wedge e^1 + \Phi_1 (e^4 \wedge e^3 + e^2 \wedge e^1) - \Phi_2 e^3 \wedge e^2 + c.c. \quad (4.35)$$

The components of the energy-momentum tensor corresponding to the field are just products of the respective field coefficients (A14) via (A7). Their structure implies immediately that $T_{\mu\nu} \ell^\mu \ell^\nu \geq 0$. Whence from the Raychaudhuri equation it follows that Φ_{00} vanishes on Δ , so does

$$\Phi_0|_\Delta = 0 \quad (4.36)$$

and so do all the components of $T_{\mu\nu}$ containing Φ_0

$$\Phi_{01}|_\Delta = \Phi_{02}|_\Delta = \Phi_{10}|_\Delta = \Phi_{20}|_\Delta = 0. \quad (4.37)$$

In consequence the Stronger Energy Condition II.1 holds for this kind of matter and (2.20) is satisfied at Δ automatically.

The component Φ_1 is encoded into the pullback onto Δ of $F_{\alpha\beta} - i *_{\mathcal{M}} F_{\alpha\beta}$

$$F_{ab} - i *_{\mathcal{M}} F_{ab} = \Phi_1 (e^2 \wedge e^1)_{ab} \quad (4.38)$$

The electromagnetic field $F_{\mu\nu}$ is subject to the Maxwell equations which in the null frame can be written in the form (A13). On Δ these equations reduce to

$$\Phi_0|_\Delta = 0, \quad D\Phi_1|_\Delta = 0, \quad (4.39a)$$

$$D\Phi_2|_\Delta = -\kappa^{(\ell)} \Phi_2 + (\bar{\delta} + 2\pi) \Phi_1. \quad (4.39b)$$

The values of Φ_1, Φ_2 given on the chosen initial slice $\tilde{\Delta}$ are then sufficient to determine the field $F_{\alpha\beta}$ at Δ (provided that all the necessary frame and connection components are given). Also the pull-back F_{ab} of F to Δ is determined just by Φ_1 which furthermore can be represented as a pull-back $\Phi_1 = \pi^* \hat{\Phi}_1$ of scalar $\hat{\Phi}_1$ defined on $\hat{\Delta}$.

The contribution of the Maxwell field to the frame expansion derived in section IV C 3 can be summarized as follows: since the Ricci tensor components (so the Maxwell field tensor) do not contribute to the 0th and 1st order of expansion, the set of data required to determine the expansions will be modified only for $n \geq 2$. In such case the modification can be summarized as:

Corollary IV.4. *Suppose Δ is a non-expanding horizon embedded in 4-dimensional electrovac spacetime. Let (x^A, v, r) be the Bondi-like coordinates defined in Section III A, (e_1, e_2, e_3, e_4) be a null frame defined in IV C 1 and IV C 2, and Φ_I , $I = 0, 1, 2$ be the electromagnetic field coefficients defined above. Then the value of the constant $\kappa^{(\ell)}$ and the following data defined on the initial slice Δ*

- *horizon geometry:* $m^\alpha, \pi, \mu, \lambda$,
- *electromagnetic field:* Φ_1 , and $\partial_r^k \Phi_2$, $\forall k \in \{0, \dots, n-2\}$,
- *the Weyl tensor component:* $\partial_r^k \Psi_4$ $\forall k \in \{0, \dots, n-2\}$

determines on Δ uniquely all the radial derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ of the frame components up to the order $k = n$. This data is free, it is not subject to any constraints, modulo Remark IV.2.

The proof of this corollary, being a modification to the proof of Corollary IV.3, is presented in Appendix C. As in the case of Corollary IV.3, the existence or uniqueness is not guaranteed (see the discussion below Corollary IV.3).

Corollary IV.4 and the form which the Maxwell-Einstein equations take on a NEH imply quite interesting property of the spacetime metric at the horizon. There is a well defined limit in which a given spacetime metric tensor $g_{\mu\nu}$ defines perturbatively—in terms of the expansion at a NEH Δ —a new, “would be” (that is provided that it exists) stationary

⁹ The decomposition is valid for a general Newman-Penrose null frame.

solution to the Einstein questions in *all* the orders in the transversal variable r . Indeed, we can easily determine the dependence on v of all the data listed in Corollary IV.4. In particular, the frame and rotation components are (by definition) v independent, whereas μ, λ are (due to the reduction to the horizon of (A8i, A8h)) exponential (when $\kappa^{(\ell)} \neq 0$) or linear (otherwise) in v respectively. Acting with ∂_r^n on the transversal evolution equations (A8k-A8r) (expressed in more convenient form as (C2-C4) one can show that the n th transversal derivative of metric (represented by the respective derivative of the frame components) behaves like

$$\partial_r^n g_{\alpha\beta}|_{\Delta} \sim g_{\alpha\beta}^{(n)} e^{-n\kappa^{(\ell)}v} + \dots + g_{\alpha\beta}^{(0)} \quad (4.40)$$

[27] if $\kappa^{(\ell)} \neq 0$. Since the Bondi-like variable always can be chosen in such a way that $\kappa^{(\ell)} \neq 0$, this result can be interpreted in the way, that the horizon neighborhood geometry settles down to the geometry representing a Killing horizon with ∂_v as a Killing vector. Note however, that the result does not mean that the horizon neighbourhood approaches the symmetric spacetime as (i) we do not know whether or not there is a metric tensor, solution to the Einstein equations, which satisfies the limit expansion and (ii) solutions to the characteristic initial Cauchy problem do not need to be analytic.

2. Characteristic Cauchy problem

Now, we can complete the data of Corollary IV.4 to characteristic Cauchy data. This will be the null frame version of the Cauchy data introduced in a geometric way in Section IV B. Here, we provide a formulation in terms of the null tangent frame $(m^\mu, \bar{m}^\mu, n^\mu, \ell^\mu)$, the corresponding Newman-Penrose coefficients of the connection, curvature, and the electromagnetic field. We apply the results of sections III A and IV D 1 to specify the class of the reduced Friedrich data [29, 30] corresponding to the case at hand and the NEHs in question.

As in Section IV B, in addition to a given NEH Δ , we use another null surface \mathcal{N}_{v_0} orthogonal to a slice $\tilde{\Delta}$ of Δ such that $v|_{\tilde{\Delta}} = v_0$. Now, in the Bondi-like coordinates (x^A, v, r) the null surfaces Δ and \mathcal{N}_{v_0} satisfy:

$$r|_{\Delta} = 0, \quad v|_{\mathcal{N}_{v_0}} = v_0. \quad (4.41)$$

The NEH horizon geometry and the component Φ_1 of the electromagnetic field defined on Δ , coupled to the component Ψ_4 of the Weyl tensor and Φ_2 set freely on the entire \mathcal{N}_{v_0} provide at the slice $\tilde{\Delta}$ the data of Corollary IV.4. Furthermore, they determine uniquely all the (spacetime) frame, connection, Maxwell field and Riemann tensor components at \mathcal{N}_{v_0} . The key idea of the proof is the observation that the Einstein-Maxwell equations and Bianchi identities form on \mathcal{N}_{v_0} a hierarchy of the ordinary differential equations. For the readers convenience the proof of this fact is presented in Appendix C 2. The consequence of the above observations is

Corollary IV.5. *Given a non-expanding horizon Δ and the transversal null surface \mathcal{N}_{v_0} the following data is the Friedrich reduced data [31]:*

- (i) *the surface gravity $\kappa^{(\ell)} \in \mathbb{R}$ ($= 0$ or $= 1$);*
- (ii) *on $\tilde{\Delta}_{v_0} = \Delta \cap \mathcal{N}_{v_0}$: m^α (by construction, tangent to $\tilde{\Delta}$), $\pi, \mu, \lambda, \Phi_1$;*
- (iii) *on \mathcal{N}_{v_0} : Φ_2, Ψ_4 .*

The data is freely defined modulo Remark IV.2. Given this data, in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$, the NP equations (A8d-A8i, A8j-A8r) coupled with the Einstein-Maxwell field equations (A7, A12), Maxwell evolution equations (A13), Bianchi identities (A11a-A11h), frame components evolution equations (4.31) and with the gauge choice equations (4.9, 4.11, 4.15, 4.17, 4.20, 4.27, 4.28, 4.29, 4.30) define a unique electrovac spacetime $(e_1, \dots, e_4, \Phi_0-\Phi_2, \Psi_0-\Psi_4, X, H, Z_A, m_A)$. In the spacetime defined in \mathcal{M}^{\pm} (the future/past to Δ part of the domain of dependence) by the resulting solution, Δ is a non-expanding horizon, (x^A, v, r) is an adapted Bondi-like coordinate system and $(m^\mu, \bar{m}^\mu, n^\mu, \ell^\mu)$ is a null frame of the properties of the frame (4.9-4.25). In particular, the vector field

$$n^\mu := -(\partial_r)^\mu \quad (4.42)$$

is null, satisfies

$$\nabla_n n = 0 \quad (4.43)$$

and is orthogonal to the slices

$$v = \text{const} \quad (4.44)$$

of the horizon. Also, the vector field

$$\xi^\mu := (\partial_v)^\mu \quad (4.45)$$

satisfies

$$\xi^\mu|_\Delta = \ell^\mu, \quad \mathcal{L}_n \xi = 0. \quad (4.46)$$

Therefore, the current vector fields \mathbf{n}^μ and ξ^μ coincide with the vector fields n^μ and ξ^μ induced in a neighborhood of Δ in Section III A.

This structure will be applied in the next section to identify and characterize the necessary and sufficient conditions for the existence of a timelike Killing field on the domain \mathcal{M}'^\pm .

V. ELECTROVAC KILLING HORIZON

A. The induced structures, the Bondi-like coordinates and the adapted null frame

We now restrict our interest to the situation when an electrovac spacetime \mathcal{M} admits a Killing vector field K^μ tangent to and null at a horizon Δ . We also assume that K^μ is an infinitesimal symmetry of the electromagnetic field, that is

$$\mathcal{L}_K F_{\mu\nu} = 0. \quad (5.1)$$

On the horizon

$$(K|_\Delta)^a = \ell^a \quad (5.2)$$

is an infinitesimal symmetry. We also assume, that the surface gravity of ℓ^a (necessarily constant) is not zero

$$\kappa^{(\ell)} \neq 0. \quad (5.3)$$

We will apply the general results of Sections II B 1, III C and III D as well as we will employ the adapted null frames introduced in Section IV.

If the NEH Δ geometry is invariant-generic (see Section III C) and ξ^μ is the Δ neighbourhood invariant vector field, then due to Theorem III.3 the Killing vector necessarily coincides with ξ^μ modulo a rescaling by a constant factor

$$K^\mu = \xi^\mu. \quad (5.4)$$

Otherwise, the results of Section III D apply. In either case, there are on Δ coordinates (x^A, v) such that

$$\ell^a = (\partial_v)^a. \quad (5.5)$$

We are assuming they are given and use the corresponding Bondi-like extension and the related Bondi-like coordinates (x^A, v, r) . Then, owing to Theorem III.4, the Killing vector K in the neighbourhood \mathcal{M}' necessarily is

$$K^\mu = (\partial_v)^\mu. \quad (5.6)$$

It turns out that the null frame (e_1, \dots, e_4) (4.26) adapted to the structures introduced on Δ , and the adapted Newman-Penrose framework defined in Section IV are surprisingly compatible with the Killing vector fields:

Lemma V.1. *Suppose K^μ is a Killing vector field tangent to a NEH (Δ, q_{ab}, D_c) . The components of the null frame (e_1, e_2, e_3, e_4) and the dual coframe (e^1, \dots, e^4) introduced in Section IV C are Lie dragged by K^μ , that is*

$$\mathcal{L}_K e^1 = \dots = \mathcal{L}_K e^4 = 0 = \mathcal{L}_K e_1 = \dots = \mathcal{L}_K e_4, \quad (5.7)$$

provided

$$\mathcal{L}_K e^1|_\Delta = \dots = \mathcal{L}_K e_4|_\Delta = 0. \quad (5.8)$$

Indeed, it follows from the following calculation true in the neighborhood \mathcal{M}' of Δ for every value of $\mu = 1, \dots, 4$ (no abstract index)

$$0 = \mathcal{L}_K(\nabla_{\mathbf{n}} e_\mu) = \nabla_{\mathbf{n}} \mathcal{L}_K e_\mu \quad (5.9)$$

where the first equality follows from $\nabla_{\mathbf{n}} e_\mu = 0$, whereas the second follows from

$$[\mathbf{n}, K] = 0 \quad (5.10)$$

which for a Killing vector K^μ implies that the parallel transport along the integral lines of \mathbf{n}^μ commutes with the flow of K^μ . The second equation in (5.9) combined with the initial condition

$$\mathcal{L}_K e_\mu|_\Delta = 0 \quad (5.11)$$

completes the proof.

Corollary V.2. *Consider a NEH Δ such that its neighbourhood admits a Killing vector field tangent to Δ and null thereon. If Δ is invariant-generic introduce on Δ coordinates (x^A, v) such that ∂_v is the invariant vector on Δ . Otherwise, assume that K^μ is not zero restricted to any null generator of Δ and introduce on Δ coordinates (x^A, v) such that*

$$K^\mu|_\Delta = (\partial_v)^\mu. \quad (5.12)$$

Extend (x^A, v) to the Bondi-like coordinates in the neighbourhood \mathcal{M}' . Introduce the null frame of Section IV C 1 and Section IV C 2.

Then, in all the \mathcal{M}' the Killing vector K^μ is of the form

$$K^\mu = (\partial_v)^\mu. \quad (5.13)$$

Furthermore, as a consequence of Lemma V.1, all the frame coefficients $e_\mu^A, e_\mu^v, e_\mu^r$ as well as all the Newman-Penrose coefficients of the Levi-Civita connection, the Maxwell field and the Weyl tensor are constant along the orbits of K^μ , that is they are independent of the variable v .

B. Necessary conditions: data at Δ , and the metric expansion

The expansion in the radial coordinate¹⁰ r at the given NEH Δ of the coefficients of the frame e_1, \dots, e_4 in the general case assuming the vacuum Einstein-Maxwell equations was developed in Section IV D 1. The presence of the Killing field imposes new constraints on the data considered on the horizon Δ following from the substitution of the symmetry condition

$$\ell^\alpha \partial_\alpha (n^\beta \partial_\beta)^k f = 0, \quad (5.14)$$

where f is any component of the frame $e_1^\alpha, \dots, e_4^\alpha$ in the Bondi-like coordinates (x^A, v, r) , and any component (in that frame) of the Levi-Civita connection, the Weyl tensor, and the Maxwell field, and $k \in \mathbb{N}$. Indeed, the equation (4.18) with 0 substituted for $\partial_v \mu$ and $\partial_v \lambda$ determines the expansion and shear (μ, λ) of n as functionals of the remaining elements of the horizon geometry—the complex vector m^a , the surface gravity $\kappa^{(\ell)}$ and the component π of the rotation 1-form potential—and the component Φ_1 of the electromagnetic field,

$$\mu = \frac{1}{\kappa^{(\ell)}} [m^a \partial_a + |\pi|^2 - 2\pi \bar{a} + \Psi_2] , \quad (5.15a)$$

$$\lambda = \frac{1}{\kappa^{(\ell)}} [\bar{m}^a \partial_a + |\pi|^2 + 2a\pi] . \quad (5.15b)$$

Also the Maxwell field equation (4.39b) upon the assumption $\partial_v \Phi_2 = 0$ determines the value of Φ_2 at Δ for known (Φ_0, Φ_1) and connection coefficients,

$$\Phi_2 = \frac{1}{\kappa^{(\ell)}} [\bar{m}^a \partial_a + 2\pi \Phi_1] . \quad (5.16)$$

¹⁰ The affine parameter of transversal null geodesics.

Hence the value of a whole Energy-Momentum tensor at Δ is known. Therefore given $(m, \kappa^{(\ell)}, \pi, \Phi_1)$ one can calculate all the connection coefficients as well as the Weyl tensor components $\Psi_0, \Psi_1, \Psi_2, \Psi_3$ (see the analysis in Section IV D 1). In the analogous way the last component Ψ_4 is determined as a functional of $(m, \kappa^{(\ell)}, \pi, \Phi_1)$ via the substitution of 0 for $\partial_v \Psi_4$ in (4.25) and expressing $\Phi_{20,r}$ therein by a suitable functional of $(m, \kappa^{(\ell)}, \pi, \Phi_1)$ following from $\Phi_0 = 0$ and the Maxwell equation (A13c),

$$\Psi_4 = \frac{1}{2\kappa^{(\ell)}} [\bar{m}^a \partial_a \Psi_3 - 3\lambda \Psi_3 + (5\pi + 2a)\Psi_3 - \Phi_{20,r} + \bar{m}^a \partial_a \Phi_{21} + (\pi + 2a)\Phi_{21} - 2\lambda \Phi_{11}] \quad (5.17)$$

In this way the free degrees of freedom (represented by the triple (m^a, π, Φ_1) defined on the horizon Δ) determine then the frame expansion up to 2nd order. Furthermore, given the frame e_1^μ, \dots, e_4^μ at Δ and the derivatives $\partial_r^k e_1^\mu, \dots, \partial_r^k e_4^\mu$ for $k = 1, \dots, n$, the values of $\partial_r^{n-1} \Phi_2$ and $\partial_r^{n-1} \Psi_4$ necessary for the $n+1$ th order of expansion can be derived by differentiating the equations (A13b) and (A11d) in the radial direction respectively (see Appendices B 1 and C 1 for the details). Finally the following is true:

Corollary V.3. *Suppose (Δ, ℓ) is a Killing horizon in a 4-dimensional electrovac spacetime and the surface gravity $\kappa^{(\ell)} \neq 0$ (5.3). Then, the following data defined on the horizon: the complex vector m^a (4.27a), the rotation 1-form potential ω_a (2.12), and the pull-back onto Δ of $(F - i *_{\mathcal{M}} F)_{ab}$, uniquely determine all the derivatives $\partial_r^n e^\mu_\nu$ of the frame (4.27) coefficients as well as all the derivatives $\partial_r^n \Phi_I$, $I = 0, 1, 2$ of the components of the electromagnetic field (A12) at the horizon Δ , for all $n \in \mathbb{N}$.*

In particular, in the Einstein vacuum case, that is in the absence of the Maxwell field, is a NEH Δ is a Killing horizon, and the Killing vector field is given at the horizon as ℓ , then all the transversal derivatives $(\mathcal{L}_n)^k g_{\mu\nu}$, $k \in \mathbb{N}$, where n is a transversal to Δ vector field, are determined by the degenerate metric tensor q_{ab} induced in Δ and the rotation 1-form potential ω_a (2.12) provided $\ell^a \omega_a \neq 0$.

From Corollary V.3 it follows immediately that, provided the spacetime metric $g_{\mu\nu}$ and the electromagnetic field tensor $F_{\mu\nu}$ are analytic, they are uniquely determined in the space-time by the data $(m, \kappa^{(\ell)}, \pi, \Phi_1)$. In a vacuum case the analyticity is ensured in the region where KVF is timelike [32, 33] and (3.4). This is the case outside (in direction against n^μ) (when $\kappa^{(k)} > 0$) or inside (for $\kappa^{(k)} < 0$) the horizon in the connected region¹¹. However, we still do not know whether the metric is analytic up to the horizon Δ .

This requirement is satisfied in particular by non-degenerate Killing horizons in the static vacuum spacetime [34]. The listed data representing a non-rotating horizon¹² uniquely determines then the metric and Maxwell field of a static spacetime in the connected region in which the Killing field is timelike.

C. Necessary conditions: data on the transversal surface \mathcal{N}_{v_0}

In subsection V B we characterized the Δ part of the characteristic Cauchy data of Section IV D 2. Now, we turn to the data defined on the null surface \mathcal{N}_{v_0} transversal to the horizon Δ which data consist of the functions (Φ_2, Ψ_4) . The presence of the Killing field inducing the null symmetry at the horizon imposes some constraints on these (otherwise free) data. The specifics of the construction of the Bondi-like coordinates imply that (see Sec. III C), given coordinates (x^A, v, r) such that the Killing field (if present) null at the horizon Δ has the form $K|_\Delta = \partial_v$, it takes this form (i.e. $K^\mu = (\partial_v)^\mu$) in the neighborhood \mathcal{M}' covered by the coordinates (see Theorem III.3 (i)). If the horizon is invariant-generic, then the coordinate v is a priori given as the invariant one and $(\partial_v)^\mu = \zeta^\mu$, the invariant vector field of the neighbourhood of Δ . Otherwise, the problem reduces to finding a suitable v on Δ . Therefore, probing for a Killing vector field reduces to the question whether the field ∂_v is a symmetry of $g_{\mu\nu}$ and $F_{\mu\nu}$ at the horizon neighbourhood. We also remember from Corollary V.2, that if ∂_v is a KVF then it preserves all the frame coefficients. The converse statement is straightforward: the independence of v of all the frame, Maxwell field, connection and Weyl tensor coefficients is equivalent to the fact that ∂_v is a KVF. The condition

$$\partial_v \Psi_4|_{\mathcal{N}_{v_0}} = 0, \quad \partial_v \Phi_2|_{\mathcal{N}_{v_0}} = 0 \quad (5.18)$$

is then a necessary condition for ∂_v to be the KVF.

Using the Bianchi identity (A11d) and the Maxwell field equation (A13b) [and expressing the operator $D := \ell^\mu \partial_\mu$ in the Bondi-like coordinate system via (4.27)] one can rewrite these conditions as the differential constraints involving

¹¹ We assume here that the edge of this region has non-empty intersection with the horizon

¹² The rotation of a static Killing horizon necessarily vanishes.

the derivatives in the directions tangent to \mathcal{N}_{v_0} only

$$(H\partial_r - \bar{X}\delta - X\bar{\delta})\Psi_4 = -(4\epsilon - \rho)\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 - \kappa_0\bar{\mu}\Phi_2\bar{\Phi}_0 \\ + \kappa_0((\pi + 2a)\Phi_2\bar{\Phi}_1 - 2\lambda\Phi_1\bar{\Phi}_1 + \partial_r\Phi_2\bar{\Phi}_0 + \bar{\sigma}\Psi_2\bar{\Phi}_2) , \quad (5.19a)$$

$$(H\partial_r - \bar{X}\delta - X\bar{\delta})\Phi_2 = \bar{\delta}\Phi_1 - \lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 . \quad (5.19b)$$

The conditions (5.18) became then the well defined on \mathcal{N}_{v_0} constraint (5.19) for the geometry components. Note that this system involves the transversal to Δ derivatives ($\partial_r\Psi_4, \partial_r\Phi_2$). It can be then treated as the completion of the 'evolution' equations (C2-C6). However, we do not know whether the resulting system of equations has a well defined Cauchy problem on Δ . The first difficulty is, that the r -dependent coefficient H vanishes at Δ . Secondly, the completed system constitutes now a system of partial differential equations (PDE's) instead of the ordinary ones. The structure of this system is not manifest, however the action of the Killing flow allows to recast it into the system defined on the Cauchy surface where an equivalent system is elliptic in the region where the KVF is timelike [33]. Unlike in the static case [34] the question about the ellipticity of the system *at* the horizon remains open.

D. The necessary and sufficient conditions

In this subsection we formulate the set of necessary and sufficient conditions for the existence of a Killing vector tangent to and null at horizon in the 4-dimensional, electrovacuum case. As above, we will use the transversal null surface \mathcal{N}_{v_0} .

Theorem V.4. *Suppose (Δ, q_{ab}, D_a) is an invariant-generic non-expanding horizon contained in 4-dimensional space-time $(\mathcal{M}, g_{\alpha\beta})$ which satisfies the vacuum Einstein-Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let ζ^μ be the invariant vector field of the neighbourhood of Δ and (x^A, v, r) be the invariant Bondi-like coordinate system (see Section III A). Each of the conditions (i) and (ii) below is equivalent to the local existence (in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$ in the point (ii) below) of a Killing vector K^μ tangent to and null at Δ , and such that $\mathcal{L}_K T_{\mu\nu} = 0$:*

$$(i) \quad \mathcal{L}_\zeta g_{\mu\nu} = 0 , \quad \mathcal{L}_\zeta T_{\mu\nu} = 0 , \quad (5.20)$$

(ii) *The Cauchy data defined on Δ and \mathcal{N}_{v_0} satisfy:*

(a) *on Δ : ζ^a is an infinitesimal symmetry of (q_{ab}, D_a) and $\mathcal{L}_\zeta F_{\mu\nu} = 0$*

(b) *on \mathcal{N}_{v_0} : the conditions (5.19) are satisfied, provided (for the necessary condition) the null frame (4.27) is constructed such that $\mathcal{L}_\zeta e_1|_\Delta = \dots = \mathcal{L}_\zeta e_4|_\Delta = 0$.*

Remark V.5. *The condition (ii)(a) above is equivalent to (4.12, 4.14, 4.17, 4.23, 4.39a, 5.15).*

The necessary conditions follow from the fact, that in the invariant Bondi-like coordinate system the invariant vector ζ of the neighborhood of Δ has the form $\zeta^\mu = (\partial_v)^\mu$, from Theorem III.3, and from the previous section. To complete the proof of sufficiency, it is enough to show that at $\Delta \cup \mathcal{N}_{v_0}$ the vector field ζ^μ satisfies the Racz conditions [35]

$$\mathcal{L}_\zeta g_{\mu\nu} = \nabla_\alpha \mathcal{L}_\zeta g_{\mu\nu} = \mathcal{L}_\zeta T_{\mu\nu} = 0 , \quad (5.21a)$$

$$\nabla^\mu \nabla_\mu \zeta_\nu + {}^{(4)}\mathcal{R}_\nu{}^\mu \zeta_\mu = 0 . \quad (5.21b)$$

These conditions are ensured by the Lemma V.6 (cond. (5.21a)) and Lemma V.7 (cond. (5.21b)) below. Once they hold, ζ is necessarily a Killing field by Racz theorem [35] (which we quote in Appendix D 1).

Lemma V.6. *Suppose (Δ, q_{ab}, D_a) is a non-expanding horizon contained in 4-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ which satisfies the vacuum Einstein-Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let (x^A, v, r) be the Bondi-like coordinate system of Section III A. Suppose the conditions (ii)(a),(b) of Theorem V.4 are satisfied (however, ∂_v is not assumed here to be the invariant vector field). Then, the vector field ∂_v satisfies at $\Delta \cup \mathcal{N}_{v_0}$ the following condition for arbitrary $n \in \mathbb{N}$,*

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \mathcal{L}_{\partial_v} g_{\mu\nu}|_\Sigma = 0 . \quad (5.22)$$

Lemma V.7. *Suppose a NEH (Δ, q_{ab}, D_c) and a vector field ∂_v satisfy all the assumptions of Lemma V.6. Then the solution to the initial value problem*

$$\nabla^\mu \nabla_\mu K'_\alpha + {}^{(4)}\mathcal{R}_\alpha{}^\mu K'_\mu = 0, \quad K'_\mu|_{\Delta \cup \mathcal{N}} = \zeta_\mu \quad (5.23)$$

agrees at $\Sigma = \Delta \cup \mathcal{N}$ with the vector field ∂_v up to arbitrary order

$$\forall_{n \in \mathbb{N}} \quad \nabla_{\alpha_1, \dots, \alpha_n}^{(n)} K'_\mu = \nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \zeta_\mu. \quad (5.24)$$

The proofs of the above Lemmas V.6 and V.7 is presented in the Appendices D 2 and D 3 respectively.

In Theorem V.4 we assumed that Δ was invariant-generic. Owing to that assumption and to Theorem III.3 the only candidate for the Killing vector field was the invariant vector ζ^μ which in the invariant Bondi-like coordinates equals ∂_v . On the other hand, if we relax the invariant-genericity assumption, we still have Theorem III D. Combined with Lemma V.6 and Lemma V.7 it leads to the following non-invariant version of Theorem V.4:

Theorem V.8. *Suppose (Δ, q_{ab}, D_a) is a NEH contained in 4-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ which satisfies the vacuum Einstein-Maxwell equations with an electromagnetic field $F_{\mu\nu}$. Let (x^A, v, r) be the Bondi-like coordinate system (see Section III A) such that $\kappa^{(\partial_v)} = \text{const} \neq 0$. Each of the conditions (i) and (ii) below is equivalent to the local existence (in the domain of dependence of $\Delta \cup \mathcal{N}_{v_0}$ in the point (ii) below) of a Killing vector K^μ tangent to Δ , such that $K|_\Delta = \partial_v$ and such that $\mathcal{L}_K T_{\mu\nu} = 0$:*

$$(i) \quad \mathcal{L}_{\partial_v} g_{\mu\nu} = 0, \quad \mathcal{L}_{\partial_v} T_{\mu\nu} = 0, \quad (5.25)$$

(ii) The Cauchy data defined on Δ and \mathcal{N}_{v_0} satisfy:

- (a) on Δ : ∂_v is an infinitesimal symmetry of (q_{ab}, D_a) and $\mathcal{L}_{\partial_v} F_{\mu\nu}|_\Delta = 0$*
- (b) on \mathcal{N}_{v_0} : the conditions (5.19) are satisfied, provided (for the necessary condition) the null frame 4.27 is constructed such that $\mathcal{L}_\zeta e_1|_\Delta = \dots = \mathcal{L}_\zeta e_4|_\Delta = 0$.*

VI. AXIAL AND HELICAL KILLING FIELDS IN 4D ELECTROVAC SPACETIME

The Killing fields null at the horizon are not the only possible type of spacetime symmetries. There are two more classes possible: axial KVF and helical KVFs. In this section we formulate the set of necessary and sufficient conditions for their existence analogous to Theorem V.4. We still assume that the studied horizons are NEHs embedded in a 4-dimensional electrovac spacetime.

A. Axial KVF

If the spacetime neighbourhood of a NEH (Δ, q_{ab}, D_c) admits a rotational Killing field Φ^μ tangent to Δ , then one can choose at Δ a null vector field ℓ^a such that

$$[\Phi, \ell] = 0, \quad \kappa^{(\ell)} = \text{const} \neq 0 \quad (6.1)$$

and a foliation of Δ by spacelike slices, each preserved by the symmetry generated by $\Phi^a|_\Delta$ [15]. In the case of an invariant-generic NEH Δ , the invariant vector ℓ^a and the invariant foliation have this property. Otherwise, we will be assuming that ℓ^a and the foliation are given. In the corresponding Bondi-like coordinates $(x^A, v, r) = (\theta, \phi, v, r)$, such that (θ, ϕ) are the spherical coordinates defined on the spheres $v = \text{const}$, $r = \text{const}$, and such that

$$\Phi^a|_\Delta = (\partial_\phi)^a \quad (6.2)$$

owing to Theorem III.3 in the invariant-generic case, and Theorem III.4 otherwise, in all the domain of the Bondi-like extension, it is true that

$$\Phi^\mu = (\partial_\phi)^\mu. \quad (6.3)$$

The Bondi-like coordinates are determined by the coordinates (θ, ϕ, v) defined on Δ by using only $\Phi|_\Delta$. We also assume, that the KVF Φ is a symmetry of the Maxwell field. Due to Lemma V.1, in the adapted null frame 4.27 such that

$$\mathcal{L}_\Phi e^1|_\Delta = \dots = \mathcal{L}_\Phi e^4|_\Delta = \mathcal{L}_\Phi e_1|_\Delta = \dots = \mathcal{L}_\Phi e_4|_\Delta = 0 \quad (6.4)$$

the Cauchy data defined on the surfaces $\Delta \cup \mathcal{N}_{v_0}$ (\mathcal{N}_{v_0} such that $v = v_0$) as specified in Corollary IV.5 is invariant with respect to $\Phi = \partial_\phi$.

The opposite statement is obviously true. Suppose the Cauchy data defined on $\Delta \cup \mathcal{N}_{v_0}$ has an infinitesimal axial symmetry ∂_ϕ . Then, the vector field Φ^μ defined in the corresponding Bondi-like coordinates as

$$\Phi^\mu = (\partial_\phi)^\mu \quad (6.5)$$

is a Killing vector field.

B. Helical KVF

If the spacetime neighbourhood of an IH admits a helical Killing vector field X (see Theorem III.4 for the definition) a result analogous to Theorem V.4 and Theorem V.8 can be established. In the current case, however, the structure of a spacetime symmetries is much richer. If present, the KVF X induces (see Theorem II.8) at Δ both, null and axial symmetry. Then, if it exists, by Theorem III.4 we construct on \mathcal{M}' the Bondi-like coordinate system, using as the boundary condition at Δ the assumption, that restriction to $T(\Delta)$ of X is a linear combination of a null ∂_v and axial ∂_ϕ symmetry generators.

Knowing the expected form of KVF one can repeat the steps performed in the proof of Theorem V.4, just inserting as a candidate for KVF the field $X^\mu := a\zeta^\mu + b(\partial_\phi)^\mu$ (where a, b are constants) instead of ζ^μ . The frame coefficients have then to satisfy an additional condition, namely that at the horizon they are invariant with respect to axial symmetry induced at it

$$\partial_\phi e^\mu|_\Delta = 0. \quad (6.6)$$

Upon that assumption the generalization of Theorem V.4 to the case of helical KVF is almost straightforward. The only step which require certain attention if the proof of (5.22) (part of the proof of Lemma V.6) at Δ as the evolution of higher order derivatives of $g_{\mu\nu}$ along the orbits of X is a priori not known. We know however that the components of Δ internal geometry (so all the elements of the set of (external) geometry data $\bar{\chi}$ except Ψ_4, Φ_2) are preserved by the flow of ∂_v and ∂_ϕ as well as the flow of X . Whence, applying the method used in proof of Lemma V.6 we can show, that also all the transversal derivatives (i.e. the derivatives over radial coordinate r) of all the elements of $\bar{\chi}$ are preserved by the flow of ∂_v , ∂_ϕ and X .

Note that if the spacetime metric is analytic at $\Delta \cup \mathcal{N}$ the invariance of all the transversal metric (and Maxwell field) derivatives with respect to ∂_ϕ implies immediately, that the Bondi-like extensions ζ, ∂_ϕ of the null and axial symmetry induced at Δ are Killing fields at the horizon neighbourhood. This is true also in higher dimension and for any compact topology of the horizon base space [36] It is a-priori not known whether this statement will remain true if we drop the analyticity assumption.

The necessary condition for X to be KVF

$$X^\mu \partial_\mu \Psi_4|_{\mathcal{N}} = X^\mu \partial_\mu \Phi_2|_{\mathcal{N}} = 0 \quad (6.7)$$

can be expressed in the form similar to (5.19)

$$\begin{aligned} (H\partial_r - \bar{X}\delta - X\bar{\delta} - C\partial_\phi)\Psi_4 &= -(4\epsilon - \rho)\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 - \kappa_0(\bar{\mu}\Phi_2\bar{\Phi}_0 \\ &\quad - (\pi + 2a)\Phi_2\bar{\Phi}_1) + \kappa_0(-2\lambda\Phi_1\bar{\Phi}_1 + \partial_r\Phi_2\bar{\Phi}_0 + \bar{\sigma}\Psi_2\bar{\Phi}_2), \end{aligned} \quad (6.8a)$$

$$(H\partial_r - \bar{X}\delta - X\bar{\delta} - C\partial_\phi)\Phi_2 = \bar{\delta}\Phi_1 - \lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2, \quad (6.8b)$$

where C is a constant of \mathcal{N} such that $X = c_1(\partial_v + C\partial_\phi)$ (with c_1 being a constant) at $\Delta \cap \mathcal{N}$. As a consequence one can formulate necessary and sufficient conditions for the existence of a helical HVF in the neighbourhood of the NEH in forms of the analogs of Theorems V.4 and V.8. The only difference with respect to those theorems is (i) different differential condition for data at \mathcal{N}_{v_0} , namely (5.19) is replaced with (6.8a) and (ii) additional condition at Δ : that the adapted null frame (4.27) is preserved by the flow of a cyclic symmetry of Δ induced on it owing to Theorem II.8.

VII. CONCLUSIONS

In the article we explored the possibility of constructing a well defined and convenient to use description of a non-expanding horizon spacetime neighbourhood. This goal has been achieved by a construction of a preferred coordinate system, built with use of the geometric invariant of the horizon geometry: invariant null vector field and

the foliation compatible with its flow. For the class of the horizons named here 'generic-invariant' (and covering all horizon geometries except for special non-generic cases) such structure is unique, whereas the small class of non-generic horizons may allow for several such structures on one horizon. This structure allowed to arrive to the following results true for the neighbourhood of the horizon of arbitrary dimension and arbitrary compact spatial slice (or equivalently base space) topology:

1. The distinguished coordinate system defined at non-expanding horizon has been extended to the horizon neighbourhood analogously to the construction of the Bondi coordinate system near the null SCRI: the horizon coordinates are transported along the (defined uniquely for generic-invariant horizon) geodesics generated by the null field transversal to the slices of distinguished foliation of the horizon. These coordinates are then supplemented by the affine parameter along the above mentioned geodesics. Specifically, the coordinates are determined uniquely by the distinguished null flow and foliation of the horizon via (3.2, 3.5, 3.6, 3.7).
2. The specific construction of the above coordinate system proves to be very convenient in case when the non-expanding horizon is a Killing horizon, that is there exists at the horizon neighbourhood a Killing vector field tangent to the horizon and null at it. The Killing vector field takes in specified coordinate system a particular very simple form given by (3.10).
That allows in particular to test immediately if the non-expanding horizon is a Killing horizon as one needs only to verify whether the fields of the form (3.10) are Killing fields.
3. The quasi-local version of the Hawking rigidity theorem [15], see also [36] allows to generalize the above results to the Killing fields, which are not necessarily null at the horizon. Then again, such fields take in Bondi-like coordinate system a specific form given by (3.11).

in the case of non-expanding horizons in four dimensional spacetime (but still for the arbitrary compact topology of the horizon base space) and arbitrary (up to energy condition II.1) matter content the known Newman-Penrose formalism allowed to construct on the horizon neighbourhood an invariant null frame, compatible with the horizon invariant structure and the Bondi-like coordinate system. In case of a generic-invariant horizon such frame is again defined uniquely. An application of this frame to describe the spacetime metric near the horizon led to the following result:

5. The invariant Bondi-like coordinate system allowed to define in an invariant way a radial expansion of the spacetime metric about the horizon. This and representation of the geometric data in a distinguished null frame allowed in turn to identify a free data needed to determine the expansion of the spacetime metric at the horizon up to desired order. This data is specified by the Corollary IV.3. It does not require to know the evolution equations of the matter fields present at the horizon neighbourhood.

However, if one considers specific matter content the matter field equations may induce additional constraints on the above (otherwise free) data.

The matter of studies was subsequently restricted to the horizons in 4-dimensional spacetime admitting Maxwell field only. In this case the distinguished null frame introduced before allows for a convenient representation of the Maxwell field equations. This in turn allowed to improve the result of point 5 as well as to establish new ones. In particular:

6. The Maxwell field equations coupled to Einstein-Maxwell ones allowed to reduce the set of data at the horizon required to determine the expansion of the spacetime metric up to desired order. Such expansion along the whole horizon is now determined by the appropriate data (specified in Corollary IV.4) on a single spatial slice of the horizon.
7. Known formulation of the characteristic initial value problem for Einstein-Maxwell field equations (together with Maxwell evolution equations) and known specification of the free data (so called Friedrich reduced data on the boundary surfaces: in our case the horizon and the null surface transversal to it) in terms of the components of geometry and matter field in the distinguished null frame allowed in turn to determine the necessary and sufficient conditions for the existence on the non-expanding horizon neighbourhood of the Killing vector field tangent to and null at the horizon. this condition takes a form of a differential constraint involving the (otherwise free) data on the null surface transversal to the horizon. In general electro-vacuum case it is the set of constraints (5.19), whereas in the vacuum case it is a single constraint (5.19a) with Φ_0, Φ_1, Φ_2 set to zero.

These constraints allow in particular to probe the spacetime geometries determined numerically near the horizon for stationarity. Also the found constraints allow to construct in a straightforward way an invariant quantity which vanishes when the a non-expanding horizon is a Killing horizon and constituting the measure of departure

from stationarity otherwise. It may then prove useful in description of the spacetime near black hole horizon in its final stages of evolution (like final stages of black hole merging process).

The developments presented in this paper, although being of considerable potential use by themselves, are essentially an exploration of possibilities to build and use the description of a black hole neighbourhood via natural expansion of the distinguished geometry structures of its horizon. Therefore, they are meant to be mainly a methodology example rather than a complete studies. For that reason, for example only the Maxwell field as a matter content has been considered when developing the results listed in points 6 and 7 above. These results can be easily extended to other types of matter, provided the appropriate formulation of characteristic initial value problem for any such type of matter exists and the free initial data for such problem are identified. In particular, the theorems and construction of the whole article can be extended in a straightforward way to admit a nonvanishing cosmological constant, thus opening the applications in description of black holes in asymptotically anti-DeSitter spacetimes.

Similarly, the results presented in point 5 rely heavily on the Newman-Penrose complex tetrad formalism, tailored specifically to 4-dimensional spacetimes. One can however introduce in arbitrary dimension a real orthonormal vielbain (see for example [13]) playing the same role. Rewriting the Einstein equations in such vielbain, while more involved than in dimension 4 does not pose any new qualitative challenges. As a consequence, at least the description of the radial metric expansion at the horizon can be extended in a systematic way to higher dimension.

The result of point 1 can be extended in two ways. On one hand, the invariant geometric structure of the horizon (although a bit modified) is still present on the dynamical horizon. Since to build the Bondi-like coordinate system on the horizon neighbourhood one only needs this structure and the congruence of the null geodesics transversal to the horizon, it is straightforward to introduce such description in the neighbourhood of a dynamical black hole. A similar to ours construction (based on general foliations by marginally trapped tubes) has been for example proposed in [37].

On the other hand, the specified coordinate system construction does not employ the Einstein field equations outside of the energy condition II.1, which in turn can be formulated in terms of purely geometric quantities (instead of the matter stress-energy tensor). As a consequence this construction can be extended also to modified theories of gravity, although for such extension certain care is needed to probe how the distinguished geometry structure at the horizon changes.

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Appendix A: The Newman-Penrose formalism

Here we briefly summarize the Newman-Penrose formalism, as specified in [38].

1. The NP null frame

The complex null vector frame $(e_1, \dots, e_4) = (m, \bar{m}, n, \ell)$ (where n, ℓ are real vectors and m is complex) of a four-dimensional spacetime consists the Newman-Penrose null tetrad if the following scalar products

$$g^{\mu\nu} m^\mu \bar{m}^\nu = 1 \qquad g^{\mu\nu} n^\mu \ell^\nu = -1 \qquad (\text{A1})$$

are the only nonvanishing products of the frame components. The dual frame corresponding to the tetrad will be denoted as (e^1, \dots, e^4) . In terms of this coframe components the metric tensor takes the form:

$$g_{\mu\nu} = e^1_\mu e^2_\nu + e^2_\mu e^1_\nu - e^3_\mu e^4_\nu - e^4_\mu e^3_\nu \qquad (\text{A2})$$

The torsion-free spacetime connection is determined by the 1-forms defined as follows

$$\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} \qquad de^\alpha + \Gamma^\alpha_\beta \wedge e^\beta = 0 \qquad (\text{A3})$$

which can be decomposed onto the following complex coefficients

$$-\Gamma_{14} = \sigma e^1 + \rho e^2 + \tau e^3 + \varkappa e^4 \quad -\frac{1}{2}(\Gamma_{12} + \Gamma_{34}) = \beta e^1 + \alpha e^2 + \gamma e^3 + \epsilon e^4 \quad (\text{A4a})$$

$$\Gamma_{23} = \mu e^1 + \lambda e^2 + \nu e^3 + \pi e^4 \quad (\text{A4b})$$

called the spin coefficients.

The Riemann tensor is given by the equation

$$\frac{1}{2} {}^{(4)}R^\alpha{}_{\beta\gamma\delta} e^\gamma \wedge e^\delta = d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \quad (\text{A5})$$

The Ricci and Weyl tensor

$${}^{(4)}\mathcal{R}_{\alpha\beta} := {}^{(4)}R^\gamma{}_{\alpha\gamma\beta} \quad {}^{(4)}\mathcal{R} := {}^{(4)}\mathcal{R}^\gamma{}_\gamma \quad (\text{A6a})$$

$${}^{(4)}C_{\alpha\beta\gamma\delta} := {}^{(4)}R_{\alpha\beta\gamma\delta} + \frac{1}{6} {}^{(4)}\mathcal{R}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) - \frac{1}{4}(g_{\alpha\gamma} {}^{(4)}\mathcal{R}_{\beta\delta} - g_{\beta\gamma} {}^{(4)}\mathcal{R}_{\alpha\delta} + g_{\beta\delta} {}^{(4)}\mathcal{R}_{\alpha\gamma} - g_{\alpha\delta} {}^{(4)}\mathcal{R}_{\beta\gamma}) \quad (\text{A6b})$$

is described in the formalism by the complex coefficients defined as:

$$\Psi_0 = - {}^{(4)}C_{4141} \quad \Phi_{00} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{44} \quad \Phi_{12} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{31} \quad (\text{A7a})$$

$$\Psi_1 = - {}^{(4)}C_{4341} \quad \Phi_{01} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{41} \quad \Phi_{20} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{22} \quad (\text{A7b})$$

$$\Psi_2 = - {}^{(4)}C_{4123} \quad \Phi_{02} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{11} \quad \Phi_{21} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{32} \quad (\text{A7c})$$

$$\Psi_3 = - {}^{(4)}C_{4323} \quad \Phi_{10} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{42} \quad \Phi_{22} = -\frac{1}{2} {}^{(4)}\mathcal{R}_{33} \quad (\text{A7d})$$

$$\Psi_4 = - {}^{(4)}C_{3232} \quad \Phi_{11} = -\frac{1}{4} ({}^{(4)}\mathcal{R}_{43} + {}^{(4)}\mathcal{R}_{12}) \quad \frac{1}{24} {}^{(4)}\mathcal{R} = \frac{1}{12} ({}^{(4)}\mathcal{R}_{43} - {}^{(4)}\mathcal{R}_{12}) \quad (\text{A7e})$$

The equation (A5) written in terms of the components (A4,A7) takes the form

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \varkappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01} \quad (\text{A8a})$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \frac{1}{24} {}^{(4)}\mathcal{R} \quad (\text{A8b})$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \quad (\text{A8c})$$

$$D\rho - \bar{\delta}\varkappa = (\rho^2 + \sigma\bar{\sigma}) + \rho(\epsilon + \bar{\epsilon}) - \bar{\varkappa}\tau - \varkappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \quad (\text{A8d})$$

$$D\sigma - \delta\varkappa = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - \varkappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0 \quad (\text{A8e})$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \varkappa\lambda - \bar{\varkappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10} \quad (\text{A8f})$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \varkappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \quad (\text{A8g})$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \bar{\beta}) - \nu\bar{\varkappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20} \quad (\text{A8h})$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\varkappa + \Psi_2 + \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{A8i})$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\varkappa + \Psi_2 + \Phi_{11} - \frac{1}{24} {}^{(4)}\mathcal{R} \quad (\text{A8j})$$

$$D\tau - \Delta\varkappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \varkappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01} \quad (\text{A8k})$$

$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21} \quad (\text{A8l})$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4 \quad (\text{A8m})$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22} \quad (\text{A8n})$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \quad (\text{A8o})$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \varkappa\bar{\nu} + \Phi_{02} \quad (\text{A8p})$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\varkappa - \Psi_2 - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{A8q})$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3 \quad (\text{A8r})$$

where the differential operators δ, D, Δ correspond to the null vectors:

$$\delta := m^\mu \partial_\mu \quad \Delta := n^\mu \partial_\mu \quad D := \ell^\mu \partial_\mu \quad (\text{A9})$$

The equalities (A8) are called the Newman-Penrose equations.

The commutators of the operators (A9) can be expressed by the spin coefficients;

$$[\Delta D - D\Delta] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta \quad (\text{A10a})$$

$$[\delta D - D\delta] = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \quad (\text{A10b})$$

$$[\delta\Delta - \Delta\delta] = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta \quad (\text{A10c})$$

$$[\bar{\delta}\delta - \delta\bar{\delta}] = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta \quad (\text{A10d})$$

The Bianchi identity written in terms of the coefficients defined by (A4,A7) consists the system of complex PDEs:

$$0 = -\bar{\delta}\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 - D\Phi_{01} + \delta\Phi_{00} \\ + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \quad (\text{A11a})$$

$$0 = \bar{\delta}\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} \\ - 2(\alpha + \bar{\tau})\Phi_{01} + 2\rho\Phi_{11} + \bar{\rho}\Phi_{02} - (\bar{\mu} - 2\gamma - 2\bar{\gamma})\Phi_{00} - 2\tau\Phi_{10} - \frac{1}{12}D^{(4)}\mathcal{R} \quad (\text{A11b})$$

$$0 = -\bar{\delta}\Psi_2 + D\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 - D\Phi_{21} + \delta\Phi_{20} \\ + 2(\bar{\rho} - \epsilon)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \bar{\kappa}\Phi_{22} - (2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - \frac{1}{12}\bar{\delta}^{(4)}\mathcal{R} \quad (\text{A11c})$$

$$0 = \bar{\delta}\Psi_3 - D\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 - \Delta\Phi_{20} + \bar{\delta}\Phi_{21} \\ + 2(\alpha - \bar{\tau})\Phi_{21} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} - 2\lambda\Phi_{11} - (\bar{\mu} + 2\gamma - 2\bar{\gamma})\Phi_{20} \quad (\text{A11d})$$

$$0 = -\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - D\Phi_{02} + \delta\Phi_{01} \\ + 2(\bar{\pi} - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} \quad (\text{A11e})$$

$$0 = -\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} \\ + 2(\bar{\mu} - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} + (\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + \frac{1}{12}\delta^{(4)}\mathcal{R} \quad (\text{A11f})$$

$$0 = -\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 - D\Phi_{22} + \delta\Phi_{21} \\ + 2(\bar{\pi} + \beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - \frac{1}{12}\Delta^{(4)}\mathcal{R} \quad (\text{A11g})$$

$$0 = -\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 + \Delta\Phi_{21} - \bar{\delta}\Phi_{22} \\ + 2(\bar{\mu} + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22} \quad (\text{A11h})$$

$$0 = -D\Phi_{11} + \delta\Phi_{10} + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - \frac{1}{8}D^{(4)}\mathcal{R} + (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} \\ + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} \quad (\text{A11i})$$

$$0 = -D\Phi_{12} + \delta\Phi_{11} + \bar{\delta}\Phi_{02} - \Delta\Phi_{01} - \frac{1}{8}\delta^{(4)}\mathcal{R} + (-2\alpha + 2\bar{\beta} + \pi - \bar{\tau})\Phi_{02} \\ + (\bar{\rho} + 2\rho - 2\bar{\epsilon})\Phi_{12} + 2(\bar{\pi} - \tau)\Phi_{11} + (2\gamma - 2\bar{\mu} - \mu)\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{21} - \kappa\Phi_{22} \quad (\text{A11j})$$

$$0 = -D\Phi_{22} + \delta\Phi_{21} + \bar{\delta}\Phi_{12} - \Delta\Phi_{11} - \frac{1}{8}\Delta^{(4)}\mathcal{R} + (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} \\ + (2\bar{\beta} + 2\pi - \bar{\tau})\Phi_{12} + (2\beta + 2\bar{\pi} - \tau)\Phi_{21} - 2(\mu + \bar{\mu})\Phi_{11} + \nu\Phi_{01} + \bar{\nu}\Phi_{10} - \bar{\lambda}\Phi_{20} - \lambda\Phi_{02} \quad (\text{A11k})$$

2. The Einstein-Maxwell field equations

Given an electromagnetic field 2-form $F_{\mu\nu}$ we define the following complex coefficients

$$\Phi_0 := F_{41} \quad \Phi_1 := \frac{1}{2}(F_{43} + F_{21}) \quad \Phi_2 := F_{23} \quad (\text{A12})$$

which completely determine it.

The Maxwell field equations expressed in terms of the field and spin coefficients take the form:

$$D\Phi_1 - \bar{\delta}\Phi_0 = (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \quad (\text{A13a})$$

$$D\Phi_2 - \bar{\delta}\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \quad (\text{A13b})$$

$$\delta\Phi_1 - \Delta\Phi_0 = (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \quad (\text{A13c})$$

$$\delta\Phi_2 - \Delta\Phi_1 = -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 \quad (\text{A13d})$$

In an electrovac spacetime with vanishing cosmological constant the Ricci tensor is related with the field energy-momentum via the Einstein field equations of the following form:

$$\Phi_{\tilde{\alpha}\tilde{\beta}} = 16\pi G \Phi_{\tilde{\alpha}} \bar{\Phi}_{\tilde{\beta}} \quad \tilde{\alpha}, \tilde{\beta} \in \{0, 1, 2\} \quad (\text{A14})$$

Appendix B: 4-dimensional neighbourhood of the horizon

In this section we present the detailed proof of the Corollary IV.3 of section IV C 3.

1. Proof of corollary IV.3

The proof is based on an explicit construction of the algorithm which allows to calculate the derivatives $\partial^n e^\mu$ of the frame components m_A, Z_A, H , provided the results for all the lower orders are given. This algorithm allows thus to establish the conclusion via induction.

In general the higher radial derivatives of the frame components at the horizon can be obtained by differentiation (of appropriate order) over r of the identity (4.25) and transversal parts (i.e. the contraction of the considered equations with n) of (A3b, A5). In particular the 0th order is given directly by (4.28), whereas the first radial derivatives of e^μ_ν are determined via (4.32) by (q, D) . To demonstrate the method for $n > 1$ we start with an explicit calculation of the 2nd order before presenting the general derivation of $n + 1$ st order.

The presented method is applicable to any kind of matter field, however here we assume that for each order of the expansion the required Ricci tensor components and their radial derivatives are given on Δ . That assumption is true for example in the Maxwell and/or scalar /and/or dilaton case where the necessary Ricci tensor components are determined via the matter field equations by the respective data defined on the initial slice $\tilde{\Delta}$. The explicit expansion in the Einstein-Maxwell case was provided in section IV D 1.

a. The 2nd order

Assume now that the geometry (q, D) and results of the first order evaluation are at our disposal. Then, acting with ∂_r on (4.31) one can derive $H_{,rr}, X_{,rr}, Z_{A,rr}, m_{A,rr}$ in terms of the first radial derivatives of the connection coefficients, which in turn are given by the equations (A8j, A8n, A8m, A8l). The resulting formula for the second radial derivatives of the frame coefficients on \mathcal{M}' reads

$$H_{,rr} = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} + (\bar{\Psi}_3 + \Phi_{21})X + (\bar{\Psi}_3 + \Phi_{12})\bar{X} \quad (\text{B1a})$$

$$X_{,rr} = -\bar{\Psi}_3 - \Phi_{12} - \Phi_{22}X - \Psi_4\bar{X} \quad (\text{B1b})$$

$$Z_{A,rr} = (\bar{\Psi}_3 + \Phi_{21})m_A + (\bar{\Psi}_3 + \Phi_{12})\bar{m}_A \quad (\text{B1c})$$

$$m_{A,rr} = -\Phi_{22}m_A - \bar{\Psi}_4\bar{m}_A. \quad (\text{B1d})$$

The values of these derivatives at the horizon are given by substituting the frame coefficients with their values on Δ . In particular:

$$H_{,rr}|_\Delta = \bar{\Psi}_2 + \Psi_2 + 2\Phi_{11} - \frac{1}{12} {}^{(4)}\mathcal{R} \quad (\text{B2a})$$

$$X_{,rr}|_\Delta = -\bar{\Psi}_3 - \Phi_{12}, \quad (\text{B2b})$$

Note that the derivatives $H_{,rr}, X_{,rr}, Z_{A,rr}$ on Δ are determined directly by (q, D) and the Ricci tensor. The last derivative, $m_{A,rr}$, involves the solution Ψ_4 to the equation

$$D\Psi_4|_\Delta = -2\kappa^{(\ell)}\Psi_4 + \bar{\delta}\Psi_3 + (5\pi + 2a)\Psi_3 - 3\lambda\Psi_2 - \bar{\mu}\Phi_{20} + 2\alpha\Phi_{21} - 2\lambda\Phi_{11} - \Phi_{20,r} \quad (\text{B3})$$

(which is the restriction to Δ of the equation (4.25)). The value of this solution is uniquely determined by the value of Ψ_4 on chosen section and the horizon geometry.

To summarize, by direct inspection of the system of equations used here we see, that the data which is not determined, thus must be specified, consists of the following components:

- (i) $\Phi_{21}, \Phi_{22}, {}^{(4)}\mathcal{R}, \Phi_{20,r}$ given on the entire Δ , and
- (ii) Ψ_4 given on an initial slice $\tilde{\Delta}$.

b. The $n + 1$ th order

In this step we assume that at our disposal are the results of the derivation up to n th order, that is:

- (i) the components of the frame and their radial derivatives up to the n th order,
- (ii) the components of the connection and their radial derivatives up to the $n - 1$ st order,
- (iii) the components of the Ricci tensor, and their derivatives up to the $n - 2$ nd order, as well as the following higher derivatives $\partial_r^{n-1}\Phi_{00}, \partial_r^{n-1}\Phi_{01}, \partial_r^{n-1}\Phi_{20}, \partial_r^{n-1}(\Phi_{11} - \frac{1}{8}{}^{(4)}\mathcal{R})$,
- (iv) all the components of the Weyl tensor and their derivatives up to the $n - 2$ nd order.

The following higher derivatives: $\partial_r^{n-1}\Psi_0, \partial_r^{n-1}\Psi_1, \partial_r^{n-1}(\Psi_2 + \frac{1}{12}{}^{(4)}\mathcal{R}), \partial_r^{n-1}(\Psi_3 - \Phi_{21})$ are then determined by the Bianchi identities (A11e-A11h). Furthermore, the $n + 1$ th radial derivatives of the frame components are given by differentiating the equations (B1), whereas the n th derivatives of the connection coefficients are given by differentiating the equations (A8k-A8r). These data are also sufficient to derive the n th derivatives of $\Psi_0, \dots, \Psi_3, \Phi_{00}, \Phi_{10}, \Phi_{20}$ via the equations obtained by differentiating (A8a-A8c, A8d-A8i)¹³ sufficiently many times.

The remaining Weyl tensor component $\partial_r^{n-1}\Psi_4$ is given by the equation derived by the differentiation of the Bianchi identity (A11d). The equation has the following form

$$D \partial_r^{n-1}\Psi_4 = -(n+1)\kappa^{(\ell)}\partial_r^{n-1}\Psi_4 + \mathcal{P}_n(e, \Gamma^{(n-1)}, \Psi^{(n-1)}, \partial_r^n\Phi_{20}, \Phi^{(n-1)}, \partial_r^{(4)}\mathcal{R}), \quad (\text{B4})$$

where e represents the components of the frame, whereas $\Gamma^{(n-1)}, \Phi^{(n-1)}$ and $\Psi^{(n-1)}$ stands for the components and their derivatives up to the order of $n - 1$ of, respectively, the connection, traceless part of the Ricci tensor, and all of the Weyl tensor except Ψ_4 .

In summary, given the results up to n th order we need to specify the following data:

- (i) $\partial_r^{n-1}\Phi_{11}, \partial_r^{n-1}\Phi_{21}, \partial_r^{n-1}\Phi_{22}, \partial_r^{n-1}{}^{(4)}\mathcal{R}, \partial_r^{n-1}\Phi_{02}$ given on the entire Δ , and
- (ii) $\partial_r^{n-1}\Psi_4$ given on the initial slice $\tilde{\Delta}$.

to determine the $n + 1$ st order of the expansion.

Appendix C: 4-dimensional electrovac NEH

This appendix contains the derivation of technical results used in section IV D: the proof of Corollary IV.4 and the detailed description of the derivation of Friedrich reduced data at the transversal surface \mathcal{N} used in section IV D 2.

1. Proof of corollary IV.4

The structure of the proof is analogous to the one presented in section B 1, that is we again construct an algorithm of derivation of $n + 1$ st order expansion for given all lower orders and use the mathematical induction. The only difference is that now part of previously undetermined data can be now determined by Maxwell field equations. The modifications to the proof of section B 1 go as follows:

- The Ricci tensor components (so the Maxwell field tensor) doesn't contribute to the 0th and 1st order of expansion, so for these orders we can directly apply analogous part of the proof in section B 1.
- In the 2nd order the components $\Phi_{11}, \Phi_{12}, \Phi_{22}$ are determined by Φ_1, Φ_2 given on the distinguished initial slice $\tilde{\Delta}$. The value of $\partial_r\Phi_{02}$ (needed to derive Ψ_4 via (B3)) at the horizon is determined by (A13c)

$$\partial_r\Phi_{20}|_{\Delta} = -8\pi G\Phi_2\bar{\delta}\bar{\Phi}_1. \quad (\text{C1})$$

- Finally, given the frame and the Maxwell field expanded to an n th order and $\partial_r^{n+1}\Phi_2|_{\tilde{\Delta}}$ (where $\tilde{\Delta}$ is a slice from the previous point) the derivative $\partial_r^{n+1}\Phi_2$ on Δ is the solution to the equation obtained by action of ∂_r^{n+1} on (A13b). This completes the set of the data needed for calculation of the $n + 1$ order of the expansion.

¹³ provided the rest of the Ricci components appearing in the equations is given

2. Characteristic IVP: data derivation on \mathcal{N}

Let \mathcal{N} be a transversal null surface defined as in section III and intersecting a non-expanding horizon Δ at the slice $\tilde{\Delta}$. The part of Friedrich reduced data in characteristic initial value problem which corresponds to \mathcal{N} consists of the following components: Newman-Penrose frame and connection coefficients, Weyl tensor components Ψ_1, \dots, Ψ_4 and Maxwell field tensor components Φ_1, Φ_2 . We show here that once we specify Ψ_4 and Φ_2 on \mathcal{N} all the remaining data are determined by the data at $\tilde{\Delta}$.

Indeed, provided the coefficients (Ψ_4, Φ_2) are known, the evolution equations (4.31c, 4.31d), the Newman-Penrose equations (combined with the appropriate Einstein field eqs) (A8n, A8m, A8l, A8r, A8o), the Maxwell equation (A13d) and the Bianchi identity (A11h) form on \mathcal{N} the following ODE system

$$\partial_r m^A = \bar{\lambda} \bar{m}^A + \mu m^A \quad (\text{C2a})$$

$$\partial_r(m^A Z_A) = \pi + \mu(m^A Z_A) + \bar{\lambda}(\bar{m}^A Z_A) \quad (\text{C2b})$$

$$\partial_r \mu = (\mu^2 + \lambda \bar{\lambda}) + \kappa_0 \Phi_2 \bar{\Phi}_2 \quad (\text{C2c})$$

$$\partial_r \lambda = 2\mu\lambda + \Psi_4 \quad (\text{C2d})$$

$$\partial_r \pi = \pi\mu + \bar{\pi}\lambda + (\Psi_3 + \kappa_0 \Phi_2 \bar{\Phi}_1) \quad (\text{C2e})$$

$$\partial_r \varpi = \mu\varpi - \lambda\bar{\varpi} + (\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) \quad (\text{C2f})$$

$$-\partial_r \Phi_1 = \delta \Phi_2 - 2\mu\Phi_1 + (\bar{\pi} - \varpi)\Phi_2 \quad (\text{C2g})$$

$$\begin{aligned} -\partial_r(\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) &= \delta \Psi_4 - \kappa_0 \delta \Phi_2 \bar{\Phi}_2 - 4\mu(\Psi_3 - \kappa_0 \Phi_2 \bar{\Phi}_1) \\ &\quad + 2(\bar{\pi} - \varpi)\Psi_4 - 2\kappa_0(\pi\Phi_{22} + \mu\Phi_{21} - \lambda\Phi_{12}) \end{aligned} \quad (\text{C2h})$$

for the coefficients $(m^A, m^A Z_A, \mu, \lambda, \pi, a, \phi_1, \Psi_3)$.¹⁴ This system has a unique solution for given initial data on $\tilde{\Delta}$ (which in turn is already determined by the horizon geometry, see previous section).

Known solution of (C2) can be next applied to the system formed by (A8q, A8p, A13c, A11g), (where the Bianchi identity (A11k) was used to determine the value of $D\Phi_{22}$ in (A11g))

$$\partial_r \rho = (\rho\mu + \sigma\lambda) + (\Psi_2 - \frac{1}{12}\Lambda) \quad (\text{C3a})$$

$$\partial_r \sigma = (\mu\sigma + \bar{\lambda}\rho) + \Phi_{02} \quad (\text{C3b})$$

$$\partial_r \Phi_0 = \delta \Phi_1 - \mu\Phi_0 + \sigma\Phi_2 \quad (\text{C3c})$$

$$\begin{aligned} -\partial_r(\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1) &= \delta \Psi_3 - \kappa_0 \delta \Phi_1 \bar{\Phi}_2 + (\bar{\pi} - \varpi) + \sigma\Psi_4 - 3\mu(\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1) \\ &\quad - \kappa_0(\rho\Phi_2 \bar{\Phi}_2 + (\pi - \varpi)\Phi_1 \bar{\Phi}_2 - 5\mu\Phi_1 \bar{\Phi}_1 - \lambda\Phi_0 \bar{\Phi}_2) \end{aligned} \quad (\text{C3d})$$

which is then the system of ODEs for $(\rho, \sigma, \Phi_0, \Psi_2)$. The solution to it (also unique for a given initial data on $\tilde{\Delta}$) determines furthermore the value of ϵ via (A8j)

$$\partial_r \epsilon = \pi\bar{\pi} + \frac{1}{2}(\bar{\pi}\varpi - \pi\bar{\varpi}) + (\Psi_2 + \kappa_0 \Phi_1 \bar{\Phi}_1 + \frac{1}{12}\Lambda) \quad (\text{C4})$$

so does determine the pair (X, H) via (4.31a, 4.31b).

The last remaining part of the reduced data: Ψ_1 is determined via eq. (A11f)

$$\partial_r \Psi_1 = \delta \Psi_2 - 2\mu\Psi_1 + 2\sigma\Psi_3 + \kappa_0(\partial_r \bar{\Phi}_0 \Phi_1 - \bar{\delta} \bar{\Phi}_0 \Phi_2) + 2\kappa_0(\bar{\mu} \bar{\Phi}_0 \Phi_1 - \rho \bar{\Phi}_1 \Phi_2), \quad (\text{C5})$$

where the radial derivatives of Φ_0, Φ_1 are determined via eqs. (C2g, C3c).

The Weyl tensor Ψ_0 component is not a part of Friedrich reduced data, however we also describe its evolution since it (as well as the evolution of $D\Phi_0$) is needed in section V. The analysis of it requires a little more effort than the other components, as we do not have direct analog of “radial evolution” equations (C2) for this component. We overcome this problem applying the Bianchi identity (A11e) which implies, that the transversal derivative of Ψ_0 is equal to:

$$\begin{aligned} -\partial_r \Psi_0 - \delta \Psi_1 + \kappa_0(D\Phi_0 \bar{\Phi}_2 - \delta \Phi_0 \bar{\Phi}_1) &= -\mu \Psi_0 - (\bar{\pi} - \varpi) \Psi_1 + 3\sigma \Psi_2 + \kappa_0(2(\epsilon - \bar{\epsilon}) + \bar{\rho})\Phi_0 \bar{\Phi}_2 \\ &\quad + \kappa_0((\bar{\pi} - 2\bar{a})\Phi_0 \bar{\Phi}_1) + 2\sigma\Phi_1 \bar{\Phi}_1 - 2\kappa\Phi_1 \bar{\Phi}_2 - \bar{\lambda}\Phi_0 \bar{\Phi}_0, \end{aligned} \quad (\text{C6})$$

where the component $D\Phi_2$ is determined by the Maxwell equation (A13b).

¹⁴ Note, that the functions $(m^A, m^A Z_A)$ appear in the system as coefficients of $\delta, \bar{\delta}$.

To compute the value of $D\Phi_0$ one can apply the equations involving the derivatives of the connection components along ℓ . Acting by operator D on the equation (A13c) and substituting the values $D\Phi_2, D\Phi_1, D\mu, D\sigma, D\pi$ via the equations (A13b, A13a, A8i, (A8e)) and the combination of (A8f, A8g) (to extract $D\pi$) one gets the following equation

$$-\partial_r D\Phi_0 = -\mu D\Phi_0 + \Phi_2 \Psi_0 + \mathcal{F}, \quad (\text{C7})$$

where \mathcal{F} is a functional of the Friedrich reduced data (frame, connection, field and Riemann except Ψ_0) and their derivatives along the directions tangent to \mathcal{N} up to 2nd order. Both the equations (C6, C7) form the system of ODEs which (similarly to the systems considered above) has a unique solution for given initial data $(\Psi_0, D\Phi_0)|_{\bar{\Delta}}$.¹⁵

Appendix D: 4-dimensional electrovac Killing horizon

Here we present the original form of Racz theorem of [35] and the proofs of Lemma V.7 and V.6 necessary to prove, that the necessary condition for the vector field ζ to be a Killing field (null at the horizon) are also the sufficient one.

1. Racz theorem

Consider the field K'_μ defined as the solution to the following initial value problem

$$\nabla^\mu \nabla_\mu K'_\alpha + {}^{(4)}\mathcal{R}_\alpha{}^\mu K'_\mu = 0 \quad K'_\mu|_{\Delta \cup \mathcal{N}} = K'_{0\mu}. \quad (\text{D1})$$

Theorem D.1. [35] Suppose $(M, g_{\mu\nu})$ is a spacetime equipped with a metric tensor and admitting matter fields represented by the set of tensor fields $\mathcal{T}_{(I)\alpha_1, \dots, \alpha_k}$ satisfying the a quasi-linear hyperbolic system

$$\nabla^\mu \nabla_\mu \mathcal{T}_{(I)} = \mathcal{F}_{(I)}(\mathcal{T}_{(J)}, \nabla_\nu \mathcal{T}_{(J)}, g_{\alpha\beta}) \quad (\text{D2})$$

and such that the energy-momentum tensor is a smooth function of the fields, their covariant derivatives and the metric, thus

$${}^{(4)}\mathcal{R}_{\mu\nu} = {}^{(4)}\mathcal{R}_{\mu\nu}(\mathcal{T}_{(J)}, \nabla_\nu \mathcal{T}_{(J)}, g_{\alpha\beta}). \quad (\text{D3})$$

Then on the domain of dependence of some initial hypersurface Σ (within an appropriate initial value problem) there exists a non-trivial Killing vector field K'^μ (such that $\mathcal{L}_{K'} \mathcal{T}_{(I)} = 0$) if and only if there exists a non-trivial initial data set $K'^\mu|_\Sigma$ for (D1a) which satisfies

$$0 = \mathcal{L}_{K'} g_{\mu\nu}|_\Sigma = \nabla_\alpha \mathcal{L}_{K'} g_{\mu\nu}|_\Sigma = \mathcal{L}_{K'} \mathcal{T}_{(I)}|_\Sigma. \quad (\text{D4})$$

2. Proof of Lemma V.6

Let us start with 0th order first. In the chosen null frame the symmetric tensor $A_{\mu\nu}$ such that

$$A_{\mu\nu} := \zeta_{\mu;\nu} + \zeta_{\nu;\mu}. \quad (\text{D5})$$

can be expressed at $\Delta \cup \mathcal{N}$ directly by the frame coefficients (H, X) and their first derivatives. Due to (4.32) the components A_{33}, A_{34}, A_{31} vanish identically. The rest of components is equal to

$$A_{44} = -DH + (\epsilon + \bar{\epsilon})H + \bar{\kappa}X + \kappa\bar{X} \quad (\text{D6a})$$

$$A_{14} = DX - \delta H + 2\pi H - \kappa + (\bar{\rho} - \epsilon + \bar{\epsilon})X + \sigma\bar{X} \quad (\text{D6b})$$

$$A_{11} = \delta X + 2\bar{a}X + \bar{\lambda}H - \bar{\sigma} \quad (\text{D6c})$$

$$A_{12} = \delta\bar{X} + \bar{\delta}X - 2\bar{a}\bar{X} - 2aX + 2\mu H - (\rho - \bar{\rho}) \quad (\text{D6d})$$

respectively.

¹⁵ As Δ is a NEH both the components vanish on it.

At the horizon all the components of $A_{\mu\nu}$ are identically zero, as $H = X = 0$ there. Furthermore, the constraint (5.19b) implies, that $D\Phi_2 = 0$ at $\Delta \cap \mathcal{N}$, thus (due to (4.39b)) Φ_2 is Lie-dragged by ζ at the entire horizon. Since the frame components are preserved by the flow of ℓ at Δ the field $\ell^\mu = \zeta^\mu|_\Delta$ at the horizon is a symmetry of $F_{\mu\nu}$.

An analysis of the behavior of ζ at the transversal surface \mathcal{N} requires a little bit more effort. To proceed, let us re-express the components of $A_{\mu\nu}$ listed in (D6) in terms of the commutators $[\delta, \partial_v]$. Decomposing ζ in the null frame and applying the equations (A10) one can express the commutator $[\delta, \partial_v]$ in terms of the connection coefficients and the derivatives of (X, H) . On the other hand, the same commutator is determined by the derivatives of the coefficients (m^A, Z_A) with respect to v

$$\begin{aligned} [\delta, \partial_v] &= ((\delta\bar{X}) - X\varpi + H\mu - \bar{\rho} - (\epsilon - \bar{\epsilon}))\delta + ((\delta X) + X\bar{\omega} + \bar{\lambda}H - \sigma)\bar{\delta} + ((\delta H) + X(\rho - \bar{\rho}) - \bar{\pi}H + \varkappa)\Delta \\ &= (m^A Z_A)_{,v}\partial_r - m^A_{,v}\partial_A \end{aligned} \quad (D7)$$

Comparing the expressions above with (D6) one realizes, that $A_{\mu\nu}$ can be written down in terms of the components of $[\delta, \partial_v]$

$$A_{44} = -\partial_v H + X\overline{[\delta, \partial_v]_\Delta} + \bar{X}[\delta, \partial_v]_\Delta \quad (D8a)$$

$$A_{14} = -X[\delta, \partial_v]_{\bar{\delta}} - \bar{X}\overline{[\delta, \partial_v]_{\delta}} - \partial_v X - [\delta, \partial_v]_\Delta \quad (D8b)$$

$$A_{11} = [\delta, \partial_v]_{\bar{\delta}} \quad (D8c)$$

$$A_{21} = [\delta, \partial_v]_{\delta} + \overline{[\delta, \partial_v]_{\delta}} \quad (D8d)$$

where

$$[\delta, \partial_v] =: [\delta, \partial_v]_{\delta}\delta + [\delta, \partial_v]_{\bar{\delta}}\bar{\delta} + [\delta, \partial_v]_{\Delta}\Delta. \quad (D9)$$

Due to the second equality in (D7) tensor $A_{\mu\nu}$ vanishes at \mathcal{N} if and only if the derivatives over v of the frame coefficients vanish there.

To verify whether the latter holds let us consider on \mathcal{N} the set $\dot{\chi}_{(I)}$ formed by the derivatives over v of the following data: the frame coefficients, the connection coefficients $\{\mu, \lambda, \pi, a, \rho, \sigma, \epsilon\}$, the Maxwell field and Weyl tensor components except Ψ_0 . At the intersection $\Delta \cap \mathcal{N}$ all these data vanish. On the other hand, one can build a system of the “evolution equations” for $\dot{\chi}_{(I)}$ allowing to evolve the initial values at $\Delta \cap \mathcal{N}$ along null geodesics spanning \mathcal{N} . Such system can be obtained by action of the operator ∂_v on the system (C2-C4, 4.31b). Similarly to the original system (C2-C4, 4.31b) it constitutes a hierarchy of quasilinear ODEs polynomial in the data it involves. As the operator ∂_v commutes with ∂_r, ∂_A the resulting system for $\dot{\chi}_{(I)}$ inherits the properties of the original one (C2-C4, 4.31b). In particular, for known values of the frame, connection, Maxwell field and Weyl tensor coefficients and known $\dot{\Psi}_4, \dot{\Phi}_2$ the new system forms the hierarchy of ODEs analogous to the hierarchy represented by (C2-C4, 4.31b). In consequence to solve our system one can apply the same algorithm as the one applied to (C2-C4, 4.31b). If the geometry data is known at \mathcal{N} then the system describing the evolution of $\dot{\chi}_{(I)}$ has a unique solution for given $(\dot{\Psi}_4, \dot{\Phi}_2)$. In particular, in the case when $\dot{\Psi}_4 = \dot{\Phi}_2 = 0$ at \mathcal{N} , the equations are linear and homogeneous in $\dot{\chi}_{(I)}$, thus $\dot{\chi}_{(I)} = 0$ is a unique solution. It means that $\dot{H} = \dot{X} = \dot{m}^A = \dot{Z}_A = 0$ and $\dot{\Phi}_0 = \dot{\Phi}_1 = \dot{\Phi}_2 = 0$, which implies vanishing of $A_{\mu\nu}$ and $\mathcal{L}_\zeta F_{\mu\nu}$ at \mathcal{N} .

To show (5.22) for arbitrary n , it is sufficient to show vanishing of the derivatives in directions transversal to Δ and \mathcal{N} respectively. This requirement is equivalent to the following condition

$$\partial_r^n \mathcal{L}_\zeta g_{\mu\nu}|_\Delta = 0, \quad \partial_v^n \mathcal{L}_\zeta g_{\mu\nu}|_\mathcal{N} = 0. \quad (D10)$$

As (5.22) holds at Σ for $n = 0$ the above condition is automatically satisfied at the intersection $\Delta \cap \mathcal{N}$. At Δ the satisfaction of the condition (D10) can be shown by direct inspection of the equations analogous to the ones used in the development of the metric expansion in section IV D 1 (and earlier in section IV C 3). Indeed the only coefficients in the set $\bar{\chi}$ of geometry data¹⁶ which are not automatically constant along the horizon null geodesics are Ψ_4, Φ_2 . Their derivatives over v are constant (exponential)¹⁷ along null geodesics for $\kappa^{(\ell)} \neq 0$ ($\kappa^{(\ell)} = 0$) respectively due to (B3, 4.39b) (as all the data on the right-hand side except Ψ_4, Φ_2 is constant along integral curves of ℓ). Whence, they vanish as (5.19) hold in particular at $\Delta \cap \mathcal{N}$.

Provided all the derivatives $\partial_v \partial_r^k \bar{\chi}_{(I)}$ up to n th order vanish, the derivatives of the next order of all the components of $\bar{\chi}_{(I)}$ except Ψ_4, Φ_2 also vanish, as they are determined by $\partial_v^k \bar{\chi}_{(I)}$ (where $k \in \{0, \dots, n\}$) via the respective

¹⁶ The set consists of all the components of frame, connection, Weyl tensor and Maxwell field.

¹⁷ I.e. Ψ_4, Φ_2 are exactly of the form $\Psi_4 = e^{-2\kappa^{(\ell)}v}\hat{\Psi}_4$, $\Phi_2 = e^{-\kappa^{(\ell)}v}\hat{\Phi}_2$ where $\hat{\Psi}_4, \hat{\Phi}_2$ are constant along the generators of Δ .

derivatives (namely $\partial_v \partial_r^n$) of equations (C2-C7). The similar derivatives of (B3, 4.39b) imply then the constancy (exponentiality)¹⁸ of $\partial_v \partial_r^{n+1} \Psi_4, \partial_v \partial_r^{n+1} \Phi_2$. They satisfy then $\partial_v \partial_r^{n+1} \Psi_4|_\Delta = \partial_v \partial_r^{n+1} \Phi_2|_\Delta = 0$ due to satisfaction of these conditions at $\Delta \cap \mathcal{N}$. Finally by induction $\partial_v \partial_r^n \bar{\chi}_{(\mathcal{I})} = 0$ for arbitrary n .

At the horizon the n th order transversal derivatives of $A_{\mu\nu}$ can be expressed as a functionals homogeneous in the elements of $\dot{\chi}$ and its transversal derivatives up to n th order. It was shown above that these components vanish at Δ . Whence

$$\partial_v \partial_r^n A_{\mu\nu}|_\Delta = 0. \quad (\text{D11})$$

The condition (D10) holds then at Δ , as it is satisfied at $\Delta \cap \mathcal{N}$.

To obtain an analogous result at \mathcal{N} we can apply the method used previously to verify the condition $A_{\mu\nu}|_{\mathcal{N}} = 0$. Now instead of initial value problem for $\dot{\chi}_{(\mathcal{I})}$ we need to consider IVP for $\partial_v^n \chi_{(\mathcal{I})}$ (where $\chi := \bar{\chi} \setminus \{\Psi_0\}$) analogous to it. This IVP has however the same properties as the one for $\dot{\chi}_{(\mathcal{I})}$. This statement completes the proof.

3. Proof of Lemma V.7

The equation (5.24) is already satisfied for $n = 0$ by (5.23). Suppose now it is satisfied up to n th order. Lemma V.6 implies then, that

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} (\nabla^\mu \nabla_\mu \zeta_\nu + {}^{(4)}\mathcal{R}_\nu{}^\mu \zeta_\mu) = 0 \quad (\text{D12})$$

at Σ . In consequence, due to (D1) and the inductive assumption the following holds

$$\nabla_{\alpha_1, \dots, \alpha_n}^{(n)} \nabla^\mu \nabla_\mu (K'_\nu - \zeta_\nu)|_\Sigma = 0. \quad (\text{D13})$$

Let us consider above equation at Δ first. Its contraction (in all the α_i) with \mathbf{n}^μ produces the following condition

$$(\mathcal{L}_\mathbf{n})^n \mathcal{L}_\ell \mathcal{L}_\mathbf{n} (K'_\mu - \zeta_\mu) + (\mathcal{L}_\mathbf{n})^n \mathcal{L}_\mathbf{n} \mathcal{L}_\ell (K'_\mu - \zeta_\mu) = \mathcal{F}(\mathcal{L}_\mathbf{n})^n \mathcal{L}_\mathbf{n} (K'_\mu - \zeta_\mu), \quad (\text{D14})$$

where we decomposed $g^{\mu\nu}$ in $\nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu$ via (A2) and applied the inductive assumption to express covariant derivatives as Lie ones. \mathcal{F} is a functional of metric derivatives up to $n + 2$ nd order.

Due to Jacobi identity and $\mathcal{L}_\ell \mathbf{n} = 0$ this formula can be re-expressed in the following way

$$\mathcal{L}_\ell (\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu) = \frac{1}{2} \mathcal{F}(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu). \quad (\text{D15})$$

It constitutes a linear homogeneous ODE for $(\mathcal{L}_\mathbf{n})^{n+1} K'_\mu$. As at $\Delta \cap \mathcal{N}$ $(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu) = 0$ its only solution at Δ is

$$(\mathcal{L}_\mathbf{n})^{n+1} (K'_\mu - \zeta_\mu)|_\Delta = 0. \quad (\text{D16})$$

The similar initial value problem can be formulated for $(\mathcal{L}_\ell)^{n+1} (K'_\mu - \zeta_\mu)$ at \mathcal{N} , whence

$$(\mathcal{L}_\ell)^{n+1} (K'_\mu - \zeta_\mu)|_{\mathcal{N}} = 0. \quad (\text{D17})$$

As a consequence, provided (5.24) is satisfied for $k \in \{0, \dots, n\}$, it holds also for $n + 1$. Whence, by induction (5.24) holds for arbitrary $n \in \mathbb{N}$.

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¹⁸ In the same strict sense as for Ψ_4, Φ_2

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